Pricing and Hedging Defaultable Claims.

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Abstract

We study the pricing and the hedging of claim ψ which depends of the default times of two firms A and B. In fact, we assume that, in the market, we can not buy or sell any defaultable bond from the firm B but we can trade only defaultable bond of the firm A. Our aim is then to find the best price and hedging of ψ using only bond of the firm A. Hence we solve this problem in two cases: firstly in a Markov framework using indifference price and solving a system of Hamilton Jacobi Bellman equation; and secondly, in a more general framework using the mean variance hedging approach and solving backward stochastic differential equations.

Keywords Default and Credit risk; Quadratic backward stochastic differential equations; Hamilton Jacobi Bellman; Mean variance hedging.

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Introduction

Models for pricing and hedging defaultable claim have generated a large debates by academics and practitioners during the last subprime crisis. The challenge is to modelize the expected losses of derivatives portfolio by taking account the counterparties defaults since they have been affected by the crisis and their agreement on the derivatives contracts can potentially vanish. In the literature, models for pricing defaultable securities have been pioneered by Merton [27]. His approach consists of explicitly linking the risk of firm's default and firm's value. Although this model is a good issue to understand the default risk, it is less useful in practical applications since it is too difficult to capture the dynamics of the firm's value which depends of many macroeconomics factors. In response of these difficulties, Duffie and Singleton [9] introduced the reduced form modeling which has been followed by Madan and Unal [26], Jeanblanc and Rutkowski [17] and others. In

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this approach, the main tool is the "default intensity process" which describes in short terms the instantaneous probability of default. This process combines with the recovery rate of the firm, represent the main tools necessary to manage the default risk. However, we should manage the default risk considering the financial market as a network where every default can affect another one and the propagation spread as far as the connections exist. In the literature, to deal with this correlation risk, the most popular approach is the copula. This approach consists of defining the joint distribution of the firms on the financial network considered given the marginal distribution of each firm on the network. In static framework [1], Li [25] was the first to develop this approach to modelize the joint distribution of the default times. But since, all computations are done without considering the evolution of the survey probability given available information then we can't describe the dynamics of the derivatives portfolio in this framework. In response of these limits on the static copula approach, El Karoui, Jeanblanc and Ying developped a conditional density approach [11]. An important point, in this framework is that given this density, we can compute explicitly the default intensity processes of firms in the financial market considered. We will follow this approach and work without losing any generality in the explicit case where financial network is defined only with two firms denoted by A and B. The intensity process jumps when any default occurs, this jump impacts the default of the firm and makes some correlation between them. We assume that we can not buy or sell any defaultable bond from the firm B but we can trade a defaultable bond of the firm A. We will consider two different cases for pricing and hedging a general defaultable claims ψ: the indifference pricing in Markov framework and the Mean-Variance hedging for the general cases.

In the first case, we so work in a Markov framework. Our aim is to find, using the correlation between the two firms, the indifference price of any contingent claim given the risk aversion defined by an exponential utility function. We express the indifference pricing as a optimization problem (see El Karoui and Rouge [12]) and use Kramkov and Schachermayer [21] dual approach. Then solving the dual problem, we find the solution of the indifference price. Moreover, the characterization of the optimal probability for the dual optimization problem is solved by Hamilton-Jacobi-Bellman (HJB) equations since the defaultable bond price is assumed to be a Markov process in this framework. We also find an explicit formula for the optimal strategy given explicitly in function of the the value function of our dual optimization problem.

In a second case, we have been interested in hedging in a general framework by Mean-Variance approach. We assume that we work in a general setting (not necessarily Markov), then we can not use the HJB equation to characterize the value function. Hence, we adopt the Mean Variance approach which has been introduced by Schweizer in [29] and generalized by many authors ([30], [13], [22], [8], [1], [23], [14]). Most of theses papers use martingales techniques and an important quantity in this context is the Variance Optimal Martingale Measure (VOM). The VOM, $\bar{\mathbb{P}}$, is the solution of the dual problem of minimizing the L^2-norm of the density $d\mathbb{Q}/d\mathbb{P}$, over all (signed) local martingale measure $\mathbb{Q}$ for the defaultable bond price of the firm A. If we consider the case of no jump of default, then

[1] The framework where we don't consider the evolution of the survey probability given a filtration

the bond price process of the firm A is continuous; in this case, Delbaen and Schachermayer in [7] prove the existence of an equivalent VOM $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$. Moreover the price of any contingent claim ψ is given by $\mathbb{E}^{\bar{\mathbb{P}}}(\psi)$. In Laurent and Pham [22], they found explicit characterization of the variance optimal martingale measure in terms of the value function of a suitable stochastic control problem. In the discontinuous case, when the so-called Mean-Variance Trade-off process (MVT) is deterministic, Arai [1] proved the same results. Since we work in discontinuous case and since in our case the Mean variance Variance Trade-off is not deterministic (due to the stochastic default intensity process), we can not apply the standards results. Hence our work is firstly to characterize the value process of the Mean-Variance problem and secondly make some links with the existence and the characterization of the VOM in some particular cases. However, we really don't need to prove and assume this existence to solve the problem. Indeed, we solve a system of quadratic Backward Stochastic Differential Equations (BSDE) and we characterize the solution of the problem using BSDE's solutions. The main contribution in this part is the explicit characterization of the BSDE's solutions without using the existence of the VOM. We obtain an explicit representation of each coefficients of quadratic backward stochastic differential equations with respect to the parameters asset of our model. In particular, the main BSDE coefficient will follow a quadratic growth and its solution is found in a constrained space. In a particular discontinuous filtration framework (where the parameters asset don't depend on the filtration generated by the jump), Lim [24] have reduced this constrained quadratic BSDE with jumps to a constrained quadratic BSDE without jumps and solved the BSDE. In the discontinuous filtration due to defaults events, we cannot do the same assumption since the intensity processes depend on the jumps (the default events). Using Kharroubi and Lim [18] technic, we will split the BSDE's with jumps into many continuous BSDEs with quadratic growth and we will conclude the existence of the solution using the standard result of Kobylanski [19].

Hence, the paper is structured as follow, in a first section, we will give some notations and present our model with some results relative to credit risk modeling. Then, in a second part, we will study the case of pricing and hedging defaultable contingent claim in a Markovian framework using indifference pricing. Then in the last section, we will study the pricing and hedging problems in a more general framework (not Markov) using mean variance hedging approach and solving a system of quadratic BSDEs.

1 The defaultable model

We work in the same model construction as in Bielecki and al. in [2] chapter 4. Let $T > 0$ be a fixed maturity time and denote by $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{[0,T]}, \mathbb{P})$ an underlying probability space. The filtration $\mathbb{F}$ is generated by a one dimensional Brownian motion $\widetilde{W}$. Let τ^A and τ^B be the two default times of firms A and B. Let define, for all $t \in [0,T]$:

$$H_t^A = 1_{\{\tau^A \leq t\}} \quad \text{and} \quad H_t^B = 1_{\{\tau^B \leq t\}}. \tag{1.1}$$

We define now some useful filtrations and definitions:

$$\mathcal{G}_t^A = \mathcal{F}_t \vee \mathcal{H}_t^B, \qquad \mathcal{G}_t^B = \mathcal{F}_t \vee \mathcal{H}_t^A \quad and \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^A \vee \mathcal{H}_t^B$$

where $\mathcal{H}^A$ (resp. $\mathcal{H}^B$) is the natural filtration generated by H^A (resp. H^B). We will denote by $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$, $\mathbb{G}^A := (\mathcal{G}_t^A)_{t\in[0,T]}$ and $\mathbb{G}^B := (\mathcal{G}_t^B)_{t\in[0,T]}$.

Definition 1.1 (Initial time). *Let η be a positive finite measure on $\mathbb{R}^2$. The random times τ^A and τ^B are called initial times if, for each $t \in [0,T]$, their joint conditional law given $\mathcal{F}_t$ is absolutely continuous with respect to η. Therefore, there exists a positive family $(g_t(y))_{t\in[0,T]}$ of $\mathbb{F}$-martingales such that*

$$G_t(\theta^A, \theta^B) = \mathbb{P}(\tau^A > \theta^A, \tau^B > \theta^B | \mathcal{F}_t) = \int_{\theta^A}^{+\infty} \int_{\theta^B}^{+\infty} g_t(y_1, y_2)\eta(dy_1, dy_2) \tag{1.2}$$

for each $\theta^A, \theta^B \in \mathbb{R}^+$ and $t \in [0,T]$.

Regarding this definition we make the following assumptions.

Assumption 1.1. *(Properties of the default times)*

- *Processes H^A and H^B have no common jumps: $\mathbb{P}(\tau_A = \tau_B) = 0$.*
- *The default times τ_A and τ_b are initial times.*

Hence, point 2. of the previous Assumption implies that the default time of firm A and B are correlated regarding our joint probability density g_t appearing in (1.2). We now give a representation Theorem of our defaultable model.

Theorem 1.1. *(Representation Theorem) Under Assumption 1.1, for $i \in \{A, B\}$, there exists a positive $\mathbb{G}$-adpated process λ^i, called the $\mathbb{P}$-intensity of H^i, such that the process M^i defined by*

$$M_t^i = H_t^i - \int_0^t \lambda_s^i ds,$$

is a $\mathbb{G}$-martingale. Moreover, any local martingale $\zeta = (\zeta_t)_{t\geq 0}$ admits the following decomposition: $\mathbb{P}$-a.s,

$$\zeta_t = \zeta_0 + \int_0^t Z_s dW_s + \int_0^t U_s^A dM_s^A + \int_0^t U_s^B dM_s^B, \quad \forall\, t \geq 0 \tag{1.3}$$

where Z, U^A and U^B are $\mathbb{G}$-predictable processes and W is the martingale part of the $\mathbb{G}$-semimartingale $\widetilde{W}$ in the enlarged filtration (see [15] for more details about the progressive enlargement of filtration and the characterization of the decomposition of any $\mathbb{F}$-semimartingale in the enlarged filtration $\mathbb{G}$).

Proof. The processes λ^A and λ^B are given explicitly since we assume that τ^A, τ^B are initial times and knowing our conditional law G. Moreover in Proposition 1.29, p54 [31], the author follows the proof of representation Theorem of Kusuoka (representation theorem when the default times are independent of the filtration $\mathbb{F}$) to construct the proof when default times are initial. $\square$

1.1 Dynamic of the Bond

In our model, the traded asset will be the defaultable bond D^A of the firm A. Using the decomposition (1.3), we represent the dynamics of this defaultable bond in the enlarged filtration $\mathbb{G}$ as in Corollary 5.3.2 of [2]

$$\frac{dD_t^A}{D_{t^-}^A} = \mu_t dt + \sigma_t^A dM_t^A + \sigma_t^B dM_t^B + \sigma_t dW_t \tag{1.4}$$

where $\mu, \sigma^A, \sigma^B, \sigma$ are $\mathbb{G}$-predictable bounded processes. Therefore, given an initial wealth $x \geq 0$, if we assume that investors follow an admissible strategies π, which is represented by a set $\mathcal{A}$ of predictable processes π such that

$$\mathbb{E}\left[\int_0^T \pi_s^2 ds\right] < +\infty, \tag{1.5}$$

then we can define the dynamics of the wealth process, started with an initial wealth x at time $t = 0$ and following a strategy π, $X^{x,\pi}$ based on the trading asset D^A by

$$dX_t^{x,\pi} = \pi_t \frac{dD_t^A}{D_{t^-}^A} = \pi_t \left[\mu_t dt + \sigma_t^A dM_t^A + \sigma_t^B dM_t^B + \sigma_t dW_t\right]. \tag{1.6}$$

Note that since all the coefficients in the dynamics of the wealth process are bounded, then for any $\pi \in \mathcal{A}$ we have that (1.5) implies:

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t^{x,\pi}|^2\right] < +\infty.$$

1.2 The Defaultable claim

We now introduce the concept of defaultable claim and give some explicit examples.

Definition 1.2. *A generic defaultable claim ψ with maturity $T > 0$ on two firms A and B is defined as a vector*

$$(X^A, X^B, Z^A, Z^B, \tau^A, \tau^B)$$

with maturity T such that:

- *The default time τ^i, $i \in \{A, B\}$ specifying the random time of default of the firm i and thus also the default events $\{\tau^i \leq t\}$ for every $t \in [0, T]$. It is always assumed that τ^i is strictly positive with probability 1.*
- *The promised payoff X^A, which represents the random payoff received by the owner of the claim ψ at time T, if there was no default of firm A prior to or at time T.*
- *The promised payoff X^B, which represents the random payoff received by the owner of the claim ψ at time T, if there was no default of firm B prior to or at time T.*
- *The recovery process Z^i, $i \in \{A, B\}$, which specifies the recovery payoff Z_{τ^i} received by the owner of a claim at time of default of the firm i, provided that the default occurs prior to or at maturity date T.*

We can introduce now the payoff at time T of this defaultable claim, which represents all cash flows associated with $(X^A, X^B, Z^A, Z^B, \tau^A, \tau^B)$. We will use too the notation ψ for this payoff. Formally, the payoff process ψ is defined through the formula by

$$\psi \quad = \quad X^A 1_{\{\tau^A > T\}} + X^B 1_{\{\tau^B > T\}} + \int_0^T Z_s^A dH_s^A + \int_0^T Z_s^B dH_s^B. \tag{1.7}$$

As an example, we can have a defaultable claim which only gives a terminal payoff of H^1 if no default occurs before time T. Hence we will not receive money if one of the firms make default. So our defaultable claim is given by

$$\psi = H^1_{\{\tau^A \vee \tau^B > T\}}.$$

Or we can have a defaultable claim which give a amount of money with respect to the time of default of the firm B and give a recovery amount H^3 if the firm A make default

$$\psi = H^1_{\{\tau^B > T\}} + H^2_{\{\tau^B \leq T\}} + \int_0^T H^3 dH_s^A.$$

2 Hedging defaultable claim in Markov framework

Let consider $\psi \in \mathcal{G}_T$ a bounded defaultable claim as defined in Definition 1.2, which depends on the default times τ^A of the firm A and τ^B of the firm B. Our aim is to find the best hedging and pricing of ψ with respect to the defaults times.

Assumption 2.2. *We assume that $\mu, \sigma^A, \sigma^B, \sigma$ and the intensity processes λ^A, λ^B are deterministic bounded functions of time, H^A and H^B.*

Remark 2.1. *Under Assumption 2.2, we have that (D^A, H^A, H^B) is a Markov process.*

We assume that the risk aversion of investors is given by an exponential utility function U with parameter δ which is

$$U(x) = -\exp(-\delta x).$$

Therefore, to define the indifference price or the hedging of ψ, we should solve the equation given by

$$u^\psi(x+p) = u^0(x),$$

where functions u^ψ and u^0 are defined by:

$$u^\psi(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[-\exp(-\delta(X_T^{x,\pi} - \psi))\right] \quad \text{and} \quad u^0(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[-\exp(-\delta X_T^{x,\pi})\right]. \tag{2.8}$$

2.1 The dual optimization formulation

To deal with the problem (2.8), we use the duality theory developped by Kramkov and Schachermayer in [21]. In fact this theory allow us to find the optimal wealth at the horizon time T and the optimal risk neutral probability $\mathbb{Q}^*$. In the sequel without loose of generality, we will assume that $r_t \equiv 0$. Let recall now some results about the dual theory.

Theorem 2.2. *[Kramkov and Schachermayer, Theorem 2.1 of [21]]*
Let U be a utility function which satisfies the standards assumptions and consider the optimization problem: $u(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^{x,\pi})\right]$, then the dual function of u defined by:

$$v(y) = \sup_{x>0}\{u(x) - xy\}, \quad u(x) = \inf_{y>0}\{v(y) + yx\}$$

is given by

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}\left(V\left[y\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\right) \tag{2.9}$$

where V represents the dual function of U and $\mathcal{M}^e$ represents the set of all risk neutral probability measures.
Moreover, there exists an optimal martingale measure $\mathbb{Q}^*$ which solves the dual problem and we have that the optimal wealth at time T is given by:

$$X_T^{x,\pi^*} = I\left[\nu Z_T^{\mathbb{Q}^*}\right], \text{ where } \nu \text{ is defined s.t.} \quad \mathbb{E}^{\mathbb{Q}^*}\left[X_T^{x,\pi^*}\right] = x.$$

where the function I represents the inverse function of U' and $Z_T^{\mathbb{Q}^*}$ represents the Radon Nikodym density on $\mathcal{G}_T$ of $\mathbb{Q}^*$ with respect to $\mathbb{P}$.

Now, we can apply this result to solve our optimization problem (2.8). We will resolve only the case $\psi \neq 0$. Indeed the particular case $\psi = 0$ could be obtain by these results. We obtain an analogous result of Delbaen and al. Theorem 2 in [5], given by the following proposition:

Proposition 2.1. *Let $\mathbb{Q}^*$ be the optimal risk neutral probability which solves the dual problem*

$$\inf_{\mathbb{Q}\in\mathcal{M}^e}\left[H(\mathbb{Q}|\mathbb{P}) - \delta\mathbb{E}^{\mathbb{Q}}(\psi)\right] \tag{2.10}$$

then the optimal strategy $\pi^* \in \mathcal{A}$ solution of the optimization problem (2.8) satisfies:

$$-\frac{1}{\delta}\ln\left(Z_T^{\mathbb{Q}^*}\right) + \psi = x + \frac{1}{\delta}\ln\left(\frac{y}{\delta}\right) + \int_0^T \pi_t^* dD_t^A \tag{2.11}$$

where $H(\mathbb{Q}|\mathbb{P})$ represents the entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ $\left(i.e. \quad \mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right)$ and y is a non negative constant.

Proof. The proof is based on the Theorem 2.2. First to match with assumptions of this Theorem in the case $\psi \neq 0$, we change the historical probability. Let define

$$\frac{d\mathbb{P}^\psi}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \frac{\exp(\delta\psi)}{\mathbb{E}\left[\exp(\delta\psi)\right]} \quad \text{and} \quad \widetilde{u}^\psi(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}^\psi\left[-\exp(-\delta X_T^{x,\pi})\right],$$

then setting $c = \mathbb{E}\left[\exp(\delta\psi)\right]$, we get

$$\begin{aligned} u^\psi(x) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[-\exp(-\delta(X_T^{x,\pi} - \psi))\right] = \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\psi}\left[-c\exp(-\delta X_T^{x,\pi})\right] \\ &= \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\psi}\left[\exp\left(-\delta\left(-\frac{1}{\delta}\log(c) + X_T^{x,\pi}\right)\right)\right] = \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\psi}\left[\exp\left(-\delta X_T^{x-\frac{1}{\delta}\log(c),\pi}\right)\right]. \end{aligned}$$

Hence by the definition of $\widetilde{u}^{\psi}(x)$ we obtain that $\widetilde{u}^{\psi}\left(x-\frac{1}{\delta}\ln(c)\right)=u^{\psi}(x)$. Then using the Theorem 2.2, the dual function of $\widetilde{u}^{\psi}$ is given, for all $y>0$, by:

$$\widetilde{v}^{\psi}(y)=\inf_{\mathbb{Q}\in\mathcal{M}^e}\mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}^{\psi}}\right)\right] \tag{2.12}$$

where

$$V(y)=\sup_{x>0}\{U(x)-xy\}=\sup_{x>0}\{-\exp(-\delta x)-xy\}=\frac{y}{\delta}\left[\ln\left(\frac{y}{\delta}\right)-1\right].$$

Then using this expression of $V(y)$ into (2.12) gives after calculation an explicit expression of the dual function which is

$$\widetilde{v}^{\psi}(y)=V(y)+\frac{y}{\delta}\ln(c)+\frac{y}{\delta}\inf_{\mathbb{Q}\in\mathcal{M}^e}\left[H(\mathbb{Q}|\mathbb{P})-\delta\mathbb{E}^{\mathbb{Q}}(\psi)\right].$$

Since $\mathbb{Q}^*$ is the optimal risk neutral probability which is solution of (2.10), we deduce that the optimal wealth at time T of the optimization problem (2.8) is given by

$$X_T^{x,\pi^*}=I\left[y\frac{Z_T^{\mathbb{Q}^*}}{Z_T^{\mathbb{Q}^{\psi}}}\right]$$

where y is defined such that $\mathbb{E}^{\mathbb{Q}^*}\left[X_T^{x,\pi^*}\right]=x-\frac{1}{\delta}\ln(c)$ and I is equal to $-V'$. Moreover from Owen [28], we can deduce that there exists an optimal strategy $\pi^*\in\mathcal{A}$ such that:

$$X_T^{x,\pi^*}=I\left[y\frac{Z_T^{\mathbb{Q}^*}}{Z_T^{\mathbb{Q}^{\psi}}}\right]=x-\frac{1}{\delta}\ln(c)+\int_0^T\pi_t^*dD_t^A.$$

In our case, since we work under the the case of exponential utility function with parameter δ, we have

$$I(y):=-\frac{1}{\delta}\ln\left(\frac{y}{\delta}\right).$$

We finally get that

$$x-\frac{1}{\delta}\ln(c)+\int_0^T\pi_t^*dD_t^A=-\frac{1}{\delta}\ln\left(\frac{y}{\delta}\right)-\frac{1}{\delta}\log\left(Z_T^{\mathbb{Q}^*}\right)+\psi-\frac{1}{\delta}\ln(c).$$

which concludes the proof of this proposition. □

2.2 Value function of the dual problem

In this part, we will solve the dual problem in a Markov framework. In fact, if we consider the same problem with a different set of probability measure like $\mathcal{M}^e=\mathcal{Q}$, where $\mathcal{Q}$ represents the set of all probability measure $\mathbb{Q}\ll\mathbb{P}$, then the value function is given by the entropy of ψ with a parameter δ. But since we work in a more restricted set of probability $\mathcal{M}^e$ which represents the set of all risk neutral probability, the value function is more difficult to precise. To characterize the value function, we first describe the set $\mathcal{M}^e$. Hence, let $\mathbb{Q}\in\mathcal{M}^e$ and define $Z_T^{\mathbb{Q}}$ be the Radon Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$. Consider the non negative martingale process $Z_t^{\mathbb{Q}}=\mathbb{E}\left[Z_T^{\mathbb{Q}}|\mathcal{G}_t\right]$ and using representation

Theorem 1.1 implies that there exists predictable processes ρ^A and ρ^B which take their values in $\mathcal{C} = (-1, +\infty)$ and a predictable process ρ which takes its values in $\mathbb{R}$ such that for all $t \in [0, T]$

$$dZ_t^{\mathbb{Q}} = Z_{t^-}^{\mathbb{Q}} \left(\rho_t^A dM_t^A + \rho_t^B dM_t^B + \rho_t dW_t \right).$$

Since $\mathbb{Q}$ is in $\mathcal{M}^e$, it is a risk neutral probability, then ZD^A is a local martingale. This implies by Ito's calculus the following equation:

$$\mu_t + \rho_t^A \sigma_t^A \lambda_t^A + \rho_t^B \sigma_t^B \lambda_t^B + \rho_t \sigma_t = 0. \tag{2.13}$$

Remark 2.2. *We notice that the process ρ depends explicitly to the values of ρ^A and ρ^B.*

Therefore using equation (2.13), the latter (2.10) can be formulated as find ρ^A and ρ^B which minimize:

$$\inf_{\mathbb{Q} \in \mathcal{M}^e} \mathbb{E}^{\mathbb{Q}} \left[\ln(Z_T^{\mathbb{Q}}) - \delta \psi \right]. \tag{2.14}$$

This is the Dual Problem we would like to solve. We make now an Assumption on the form of our defaultable claim ψ.

Assumption 2.3. *The defaultable claim $\psi \in \mathcal{G}_T$ is given by*

$$\psi = g(D_T^A) \mathbf{1}_{\{\tau^B > T\}} + f(D_{\tau^B -}^A) \mathbf{1}_{\{\tau^B \leq T\}}$$

where g and f are two bounded continuous functions.

Remark 2.3.
1. *We chose to take a defaultable claim which depends only to the default time of the firm B. However, we could have been take a defaultable claim which depends to the default time of the firm A too. The calculus would have been longer but the results will be the same.*
2. *Moreover, taking a defaultable claim depending only to the default time of the firm B has an economic sense. Indeed, our traded asset is the defaultable bond of the firm A, so it is justified to take payoff g and f function of D^A, therefore if we see the firm B as an insurance company which covers the firm A, then the default of B means the counterparty default risk.*

Proposition 2.2. *Under Assumption 2.3, the value function of the dual problem (2.14) is given by:*

$$V(t, D_t^A, H_t^A, H_t^B) := \inf_{\rho^A, \rho^B \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T j(s, \rho_s^A, \rho_s^B, D_s^A) ds - \delta g(D_T^A) \mathbf{1}_{\{\tau^B > T\}} \Big| D_t^A, H_t^A, H_t^B \right] \tag{2.15}$$

where the function j is defined by:

$$j(s, \rho_s^A, \rho_s^B, D_s^A) = \sum_{i \in \{A, B\}} \lambda_s^i \left[(1 + \rho_s^i) \ln(1 + \rho_s^i) - \rho_s^i \right] - \delta (1 + \rho_s^B) \lambda_s^B f(D_s^A) + \frac{1}{2} \rho_s^2. \tag{2.16}$$

Proof. The proof is based on the Itô's formula. We write first the dynamics of $\ln(Z^{\mathbb{Q}})$ under $\mathbb{Q}$ which is given by

$$d\ln(Z_t^{\mathbb{Q}}) = \sum_{i\in\{A,B\}} \rho_t^i dM_t^i + \left[\ln(1+\rho_t^i) - \rho_t^i\right] dH_t^i + \rho_t dW_t - \frac{1}{2}\rho_t^2 dt.$$

Using Girsanov theorem, the processes defined for all $i \in \{A,B\}$ by

$$\widetilde{M}_t^i = M_t^i - \int_0^t \rho_s^i \lambda_s^i ds \quad and \quad \widetilde{W}_t = W_t - \int_0^t \rho_s ds$$

are $\mathbb{Q}$-martingales. Hence we obtain that

$$\ln(Z_T^{\mathbb{Q}}) - \delta\psi = \int_0^T \sum_{i\in\{A,B\}} \lambda_t^i[(1+\rho_t^i)\ln(1+\rho_t^i) - \rho_t^i]dt - \delta\left[\int_0^T f(D_{t^-}^A)dH_t^B + g(D_T^A)(1-H_T^B)\right] + \int_0^T \frac{1}{2}\rho_t^2 dt + M_T^{\mathbb{Q}}$$

where $M^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale. Then we can rewrite the dual problem using the last expression:

$$\inf_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}^{\mathbb{Q}}\left[\ln(Z_T^{\mathbb{Q}}) - \delta\psi\right] = \inf_{\rho^A,\rho^B\in\mathcal{C}} \mathbb{E}^{\mathbb{Q}}\left[\int_0^T j(s,\rho_s^A,\rho_s^B,D_s^A)ds - \delta(1-H_T^B)g(D_T^A)\right]$$

where j is given in (2.16). Since by Remark 2.1, the process (D^A, H^A, H^B) is a Markov process, then using the standards results of [4] the value function of the dual optimization problem is given by:

$$V(t,D_t^A,H_t^A,H_t^B) = \inf_{\rho^A,\rho^B\in\mathcal{C}} \mathbb{E}^{\mathbb{Q}}\left[\int_t^T j(s,\rho_s^A,\rho_s^B,D_s^A)ds - \delta g(D_T^A)\mathbf{1}_{\{\tau^B>T\}}\middle| D_t^A,H_t^A,H_t^B\right].$$

□

We need now to evaluate an explicit form of the value function.

Proposition 2.3. *Let $z = (x,h^A,h^B)$ and $h = (h^A,h^B)$, then the value function of the dual optimization problem is solution of the following Hamilton-Jacobi-Bellman equation:*

$$\frac{\partial V}{\partial t}(t,z) + \frac{1}{2}\frac{\partial V}{\partial x^2}(t,z)\sigma^2(t,z) + \inf_{\rho^A,\rho^B\in\mathcal{C}}\{\mathcal{L}_{\rho^A,\rho^B}V(t,z) + j(t,\rho_t^A,\rho_t^B)\} = 0, \qquad V(T,z) = g(x)(1-h^B) \tag{2.17}$$

where

$$\mathcal{L}_{\rho^A,\rho^B}V(t,z) = \sum_{i\in\{A,B\}}\left[-\frac{\partial V}{\partial z}(t,z)\sigma^i(t,z) + \left(V(t,z^i) - V(t,z)\right)\right](1+\rho_t^i)\lambda^i(t,h)$$

and $z^i = \left(x(1+\sigma^i(t,z)), h^A + \alpha^i, h^B + 1 - \alpha^i\right)$ where $\alpha^A = 1$ and $\alpha^B = 0$. Moreover given the value function, the optimal strategy satisfies:

$$\pi_t^* = -\frac{1}{\delta}\left(\frac{\partial V}{\partial x}(t,z) + \frac{\bar{\rho}_t}{D_{t^-}^A\sigma(t,z)}\right)$$

where the process $\bar{\rho}$ is explicitly given with the optimal control $\bar{\rho}^i$, $i \in \{A,B\}$, see the relation (2.13).

Proof. From Proposition 2.2, we find that the value function of the dual optimization problem is given by (2.15). Since $\left(D^A, H^A, H^B\right)$ is Markovian under $\mathbb{P}$ and the risk neutral probability measure $\mathbb{Q}$ depends on the control (ρ^A, ρ^B), we can apply the same method as in [4] section 3.2 and 3.3. So, using now Hamilton-Jacobi-Bellman (HJB) equation we get:

$$V(t, D_t^A, H_t^A, H_t^B) = \inf_{\rho^A, \rho^B \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^{t+h} j(s, \rho_s^A, \rho_s^B, D_s^A) ds + V(t+h, H_{t+h}^A, H_{t+h}^B) \middle| D_t^A, H_t^A, H_t^B \right].$$

Then the value function solve the HJB equation (2.17).
We will find now the optimal strategy given the value function. Let recall that from Theorem 2.2, the optimal risk neutral probability and the value function exist. Let define $\bar{\rho}^A, \bar{\rho}^B$ and $\bar{\rho}$ the optimal density parameters. Since $\bar{\rho}^A$ and $\bar{\rho}^B$ are optimal for the HJB equation, assuming $\sigma(t,z) \neq 0$, using first order condition we find for $i \in \{A, B\}$:

$$\left[(V(t, z^i) - V(t, z)) - x\sigma^i(t, z) \frac{\partial V}{\partial x}(t, z) + \ln(1 + \bar{\rho}_t^i) - \frac{\sigma^i(t, z)}{\sigma(t, z)} \bar{\rho}_t \right] \lambda^i(t, h) = \delta(1 - \alpha^i) f(x) \lambda^i(t, h). \tag{2.18}$$

Then using the HJB equation (2.17) and the relation (2.18), we find the following relation:

$$-\frac{1}{2}\bar{\rho}_t^2 + \sum_{i\in\{A,B\}} \bar{\rho}_t^i \lambda^i(t,h) = \sum_{i\in\{A,B\}} (1+\bar{\rho}_t^i) \frac{\sigma^i(t,z)}{\sigma(t,z)} \bar{\rho}_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,z) x^2 \sigma^2(t,z) + \frac{\partial V}{\partial t}(t,z). \tag{2.19}$$

Let recall the Ito's decomposition of the process $\ln(Z^{\mathbb{Q}^*})$:

$$\ln(Z_T^{\mathbb{Q}^*}) = \int_0^T [\bar{\rho}_t d\bar{W}_t + \frac{1}{2}\bar{\rho}_t^2 dt] + \int_0^T \sum_{i\in\{A,B\}} \left[\ln(1+\bar{\rho}_t^i) dH_t^i - \bar{\rho}_t^i \lambda^i(t,h) \right].$$

Then using equations (2.18) and (2.19), we find an useful and more explicit decomposition of the process $\ln(Z_T^{\mathbb{Q}^*})$:

$$\begin{aligned}
\ln(Z_T^{\mathbb{Q}^*}) = &\int_0^T -\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,z_t)(D_{t^-}^A)^2 \sigma^2(t,z_t) dt - \int_0^T \frac{\partial V}{\partial t}(t,z_t) dt + \int_0^T \bar{\rho}_t d\bar{W}_t \\
&- \sum_{i\in\{A,B\}} \left[(V(t,z_t^i) - V(t,z_t)) - D_{t^-}^A \sigma^i(t,z_t) \frac{\partial V}{\partial x}(t,z) \right] dH_t^i \\
&+ \int_0^T \sum_{i\in\{A,B\}} \frac{\sigma^i(t,z_t)}{\sigma(t,z_t)} \bar{\rho}_t [dH_t^i - (1+\bar{\rho}_t^i)\lambda^i(t,h_t)] + \int_0^T \delta f(D_{t^-}^A) dH_t^B
\end{aligned}$$

where $z_t = (D_t^A, H_t^A, H_t^B)$ and $h_t = (H_t^A, H_t^B)$. Then using the Itô's decomposition of $V(T, D_T^A, H_T^A, H_T^B)$, we find:

$$\begin{aligned}
\ln(Z_T^{\mathbb{Q}^*}) = &\int_0^T \frac{\bar{\rho}_t}{\sigma(t,z_t)} \left[\sigma(t,z_t) d\bar{W}_t + \sum_{i\in\{A,B\}} \sigma^i(t,z_t) d\bar{M}_t^i \right] + \delta f(D_{\tau^B-}^A) \mathbf{1}_{\{\tau^B \le T\}} \\
&- V(T, D_T^A, H_T^A, H_T^B) + V(0, D_0^A, H_0^A, H_0^B) + \int_0^T \frac{\partial V}{\partial x}(t,z_t) dD_t^A.
\end{aligned}$$

Since

$$V\left(T, D_T^A, H_T^A, H_T^B\right) = -\delta g(D_T^A)(1 - H_T^B)$$

and

$$\psi = f(D^A_{\tau^B-})\mathbf{1}_{\{\tau^B \le T\}} + g(D^A_T)(1 - H^B_T)$$

we get:

$$\ln(Z_T^{\mathbb{Q}^*}) - \delta\psi = V(0, D_0^A, H_0^A, H_0^B) + \int_0^T \left[\frac{\bar{\rho}_t}{D^A_{t^-}\sigma(t,z)} + \frac{\partial V}{\partial x}(t, z_t) \right] dD_t^A.$$

From Definition of the value function, we have

$$V(0, D_0^A, H_0^A, H_0^B) = \mathbb{E}^{\mathbb{Q}^*}\left[\ln(Z_T^{\mathbb{Q}^*}) - \delta\psi\right]$$

using the fact that $\mathbb{E}^{\mathbb{Q}^*}\left[X_T^{x,\pi^*}\right] = x - \frac{1}{\delta}\ln(c)$ where $X_T^{x,\pi^*} = -\frac{1}{\delta}\ln\left(\frac{1}{\delta}\frac{Z_T^{\mathbb{Q}^*}}{Z^{\mathbb{P}^\psi}}\right)$ (see Theorem 2.2), we deduce that

$$\mathbb{E}^{\mathbb{Q}^*}\left[-\frac{1}{\delta}\ln(Z_T^{\mathbb{Q}^*}) + \psi - \frac{1}{\delta}\ln(c) - \frac{1}{\delta}\ln\left(\frac{y}{\delta}\right)\right] = x - \frac{1}{\delta}\ln(c).$$

Hence, we conclude

$$V(0, D_0^A, H_0^A, H_0^B) = -\delta x - \ln\left(\frac{y}{\delta}\right).$$

Finally, we find

$$-\frac{1}{\delta}\ln(Z_T^{\mathbb{Q}^*}) + \psi = x + \frac{1}{\delta}\ln\left(\frac{y}{\delta}\right) + \int_0^T -\frac{1}{\delta}\left[\frac{\bar{\rho}_t}{D^A_{t^-}\sigma(t,z)} + \frac{\partial V}{\partial x}(t, z_t)\right] dD_t^A.$$

Therefore from equation (2.11), we obtain the result of the Proposition. □

In conclusion, we have find that since we can characterize the optimal probability for the dual optimization problem using Kramkov and Schachermayer Theorem, we can characterize the HJB equation solution of our Dual problem and then this allows us to find the optimal strategy for the primal solution for a defaultable contingent claim ψ. Therefore we can find for $\psi = 0$ and $\psi \neq 0$, the optimal strategy in the both cases and deduce the indifference price p of a defaultable contingent claim solving the equation $u^\psi(x+p) = u^0(x)$.

3 Generalization of the hedging in a general framework: Mean-Variance approach

In this part, we assume that we work in a more general setting (not necessarily Markov), then we can not use the HJB equation to characterize the value function. To solve our problem we will use the Mean Variance approach. It is a well-known methodology to manage hedging in general case. It seems to have been introduced in 1992 by Schweizer [29]. An important quantity in this context is the Variance Optimal Martingale Measure (VOM). The VOM, $\bar{\mathbb{P}}$, is the solution of the dual problem of minimizing the L^2-norm of the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$, over all (signed) local martingale measure $\mathbb{Q}$ for D^A. Let recall now the Mean-Variance problem:

$$V(x) = \min_{\pi \in \mathcal{A}} \mathbb{E}\left[(X_T^{x,\pi} - \psi)^2\right]. \tag{3.20}$$

If we assume $\mathbb{G} = \mathbb{F}$ (in this case we do not consider jump of default), then the process D^A is continuous. In this case Delbean and Schachermayer [7] prove the existence of an equivalent VOM $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ and the fact that the price of ψ is given by $\mathbb{E}^{\bar{\mathbb{P}}}(\psi)$. In discontinuous case, when the so-called Mean-Variance Trade-off process (MVT) (see [29] for definition) is deterministic, Arai [1] prove the same results. Since we work in discontinuous case and since the Mean Variance Trade-off process is not more deterministic (due to the stochastic default intensity process), we cannot apply the standards results.

Remark 3.4. *Indeed, in this part we do not more assume that intensity processes λ^A and λ^B be deterministic. We take general stochastic default intensity processes. But we assume that default times τ^A and τ^B are ordered, $\tau^A < \tau^B$ and that the* **(H)***-hypothesis holds. A financial interpretation of this assumption could be the counterparty risk. Indeed, the firm A could be a bank (counterparty) and the firm B its company assurance which cover its default.*

So our work is firstly to characterize the value process of the Mean-Variance problem using system of BSDE's. Secondly make some links with the existence and the characterization of the VOM in some particular cases and thirdly prove the existence of the solution of each BSDE and give a verification Theorem. We begin by recalling some usual spaces:
• For $s \leq T$, $\mathcal{S}^\infty[s,T]$ is the Banach space of $\mathbb{R}$-valued cadlag processes X such that there exists a constant C satisfying

$$\|X\|_{\mathcal{S}^\infty[s,T]} := \sup_{t\in[s,T]} |X_t| \leq C < +\infty$$

• For $s \leq T$, $\mathcal{H}^2[s,T]$ is the Hilbert space of $\mathbb{R}$-valued predictable processes Z such that

$$\|Z\|_{\mathcal{H}^2[s,T]} := \left(\mathbb{E}\Big[\int_s^T |Z_t|^2\, dt\Big]\right)^{\frac{1}{2}} < +\infty$$

• BMO is the space of $\mathbb{G}$-adapted matingale such that for any stopping times $0 \leq \sigma \leq \tau \leq T$, there exists a non negative constant $c > 0$ such that:

$$\mathbb{E}\left[[M]_\tau - [M]_{\sigma^-} | \mathcal{G}_\sigma\right] \leq c.$$

when $M = Z.W \in$ BMO, to simplify notation we write $Z \in$ BMO.

Definition 3.3 ($R_2(\mathbb{P})$ condition)**.** *Let Z be a uniformtly integrable martingale with $Z_0 = 1$ and $Z_T > 0$, we say that Z satisfies reverse Hölder condition $R_2(\mathbb{P})$ under $\mathbb{P}$ if there exists a constant $c > 0$ such that for every stopping times σ, we have:*

$$\mathbb{E}\left[\left(\frac{Z_T^2}{Z_\sigma^2}\right)^2 |\mathcal{G}_\sigma\right] \leq c.$$

3.1 Characterization of the optimal cost via BSDE

On our problem of *mean-variance hedging (MVH)* (3.20), the performance of an admissible trading strategy $\pi \in \mathcal{A}$ is measured over the finite horizon T for an initial capital $x > 0$ by

$$J^\psi(T,\pi) = \mathbb{E}[(X_T^{x,\pi} - \psi)^2]. \tag{3.21}$$

We use the dynamic programming principle to solve our mean variance hedging problem. Let first denote by $\mathcal{A}(t,\nu)$ the set of controls coinciding with ν until time $t \in [0,T]$

$$\mathcal{A}(t,\nu) = \{\pi \in \mathcal{A} : \pi_{.\wedge t} = \nu_{.\wedge t}\}. \tag{3.22}$$

We can now define, for all $t \in [0,T]$, the dynamic version of (3.21) which is given by

$$J^{\psi}(t,\pi) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}(t,\nu)} \mathbb{E}\left[\left(X_T^{\psi,\pi} - \psi\right)^2 |\mathcal{G}_t\right]. \tag{3.23}$$

Let recall now the dynamic programming principle given in El Karoui [10].

Theorem 3.3. *Let $\mathcal{S}$ the set of $\mathbb{G}$-stopping times.*

1. *The family $\{J^{\psi}(\tau,\nu), \tau \in \mathcal{S}, \nu \in \mathcal{A}\}$ is a submartingale system, this implies that for any $\nu \in \mathcal{A}$, we have for any $\sigma \leq \tau$, the submartingale property:*

$$\mathbb{E}\left[J^{\psi}(\tau,\nu^0)|\mathcal{G}_{\sigma}\right] \geq J^{\psi}(\sigma,\nu), \quad \mathbb{P}-a.s \tag{3.24}$$

2. *$\nu^* \in \mathcal{A}$ is optimal if and only if $\{J^{\psi}(\tau,\nu^*), \tau \in \mathcal{S}\}$ is a martingale system, this means that instead of (3.24), we have for any stopping times $\sigma \leq \tau$:*

$$\mathbb{E}\left[J^{\psi}(\tau,\nu^*)|\mathcal{G}_{\sigma}\right] = J^{\psi}(\sigma,\nu^*), \quad \mathbb{P}-a.s$$

3. *For any $\nu \in \mathcal{A}$, there exists an adapted RCLL process $J^{\psi}(\nu) = (J^{\psi}(\nu)_t)_{0\leq t\leq T}$ which is right closed submartingale such that:*

$$J_{\tau}^{\psi}(\nu) = J^{\psi}(\tau,\nu), \mathbb{P}-a.s, \text{ for any stopping time } \tau.$$

We search as in Lim [23] a quadratic decomposition form for J_t^{ψ} as

$$J_t^{\psi}(\pi) = \Theta_t(X_t^{x,\pi} - Y_t)^2 + \xi_t \tag{3.25}$$

such that Θ is a non-negative $\mathbb{G}$-adapted process and Y, ξ are two $\mathbb{G}$-adapted processes. So, we will assume the quadratic form (3.25) of the cost conditional J^{ψ} with respect to the wealth process and use the Theorem 3.3 to characterize the triple (Θ, Y, ξ) as solution of three BSDEs. We will verify in the section 3.2 that the assumption of the quadratic decomposition form and the optimality and admissibility of the founded optimal strategy are satisfied.

So, let $\pi \in \mathcal{A}$ be an admissible strategy, by representation Theorem 1.1, we have that the triplet (Θ, Y, ξ) need to satisfies the following BSDEs:

$$\begin{aligned}
\frac{d\Theta_t}{\Theta_{t^-}} &= -g_t^1(\Theta_t,\theta_t^A,\theta_t^B,\beta_t)dt + \theta_t^A dM_t^A + \theta_t^B dM_t^B + \beta_t dW_t, && \Theta_T = 1\\
dY_t &= -g_t^2(Y_t,U_t^A,U_t^B,Z_t)dt + U_t^A dM_t^A + U_t^B dM_t^B \ + Z_t dW_t, && Y_T = \psi\\
d\xi_t &= -g_t^3(\xi_t,\epsilon_t^A,\epsilon_t^B,R_t)dt + \epsilon_t^A dM_t^A + \epsilon_t^B dM_t^B + R_t dW_t, && \xi_T = 0.
\end{aligned} \tag{3.26}$$

with the constraint that $\Theta_t \geq \delta > 0$, for some non negative constant δ, for all $t \in [0,T]$. The processes $\theta^A, \theta^B, U^A, U^B, \epsilon^A$ and ϵ^B are $\mathbb{G}$-predictable. Hence, we can use Itô's formula and integration by part for jump processes to find the decomposition of $J^\psi(\pi)$. Let recall that for any S, L semimartingale, we have that

$$d(S_t L_t) = S_{t^-} dL_t + L_{t^-} dS_t + d[S,L]_t.$$

In our framework since jump comes from defaults events we get

$$d[S,L]_t = \langle S^c, L^c \rangle_t + \sum_{i\in\{A,B\}} \Delta S_t^i \Delta L_t^i dH_t^i.$$

Applying these results for $S = L = (X^{x,\pi} - Y)$ gives:

$$\begin{aligned} d(X^{x,\pi} - Y)_t^2 &= 2(X_{t^-}^{x,\pi} - Y_{t^-}) \left[(\pi_t \mu_t + g_t^2)dt + \sum_{i\in\{A,B\}} (\pi_t \sigma_t^i - U_t^i) dM_t^i + (\pi_t \sigma_t - Z_t) dW_t \right] \\ &\quad + (\sigma_t \pi_t - Z_t)^2 dt + \sum_{i\in\{A,B\}} (\pi_t \sigma_t^i - U_t^i)^2 dH_t^i. \end{aligned}$$

Secondly take $S = \Theta$ and $L = (X^{x,\pi} - Y)^2$, let define $K := (X^{x,\pi} - Y)$, we find:

$$\begin{aligned} d\left(\Theta K^2\right)_t &= 2K_{t^-}\Theta_{t^-} \left[(\pi_t \mu_t + g_t^2)dt + \sum_{i\in\{A,B\}} (\pi_t \sigma_t^i - U_t^i) dM_t^i + (\pi_t \sigma_t - Z_t) dW_t \right] \\ &\quad + \Theta_{t^-}(\sigma_t \pi_t - Z_t)^2 dt + \sum_{i\in\{A,B\}} \Theta_{t^-}(\pi_t \sigma_t^i - U_t^i)^2 dH_t^i - \Theta_{t^-} K_{t^-}^2 g_t^1 dt \\ &\quad + \Theta_{t^-} K_{t^-}^2 \left[\sum_{i\in\{A,B\}} \theta_t^i dM_t^i + \beta_t dW_t \right] + 2K_{t^-}\Theta_{t^-}(\pi_t \sigma_t - Z_t)\beta_t dt \\ &\quad + \sum_{i\in\{A,B\}} \left[(\pi_t \sigma_t^i - U_t^i)^2 + 2K_{t^-}(\pi_t \sigma_t^i - U_t^i) \right] \theta_t^i \Theta_{t^-} dH_t^i. \end{aligned}$$

Using this decomposition, we can write explicitly the dynamics of $J^\psi(\pi)$ for any $\pi \in \mathcal{A}$, $dJ_t^\psi(\pi) = dM_t^\pi + dV_t^\pi$ where M_t^π is the martingale part and V_t^π the finite variation part of J_t^ψ:

$$dJ_t^\psi(\pi) = dM_t^\pi + \Theta_{t^-} \left[\pi_t^2 a_t + 2\pi_t(b_t K_t + c_t) + 2K_t(g_t^2 - u_t) - K_t^2 g_t^1 + v_t \right] dt - g_t^3 dt \tag{3.27}$$

where processes are defined respectively by:

$$\begin{aligned} a_t &= \sigma_t^2 + \sum_{i\in\{A,B\}} (\sigma_t^i)^2 (1+\theta_t^i)\lambda_t^i > 0, \quad b_t = \mu_t + \sigma_t \beta_t + \sum_{i\in\{A,B\}} \sigma_t^i \theta_t^i \lambda_t^i, \\ c_t &= -\sigma_t Z_t - \sum_{i\in\{A,B\}} \sigma_t^i U_t^i (1+\theta_t^i)\lambda_t^i, \quad v_t = Z_t^2 + \sum_{i\in\{A,B\}} (U_t^i)^2 (1+\theta_t^i)\lambda_t^i, \\ &\text{and} \quad u_t = \beta_t Z_t + \sum_{i\in\{A,B\}} U_t^i \theta_t^i \lambda_t^i \end{aligned} \tag{3.28}$$

Using now Theorem 3.3, we have that, for any $\pi \in \mathcal{A}$, the process $J^{\psi}(\pi)$ is a submartingale and that there exists a startegy $\pi^* \in \mathcal{A}$ such that $J^{\psi}(\pi^*)$ is a martingale. This martingale property implies that we should find π^* such that the finite variation part of $J^{\psi}(\pi^*)$ vanishes. Since the coefficients g^1, g^2 and g^3 do not depend on the strategy π, using the first order condition, we obtain

$$\pi_t^* = -\frac{b_t K_t + c_t}{a_t}, \quad t \le T \tag{3.29}$$

where $K_t = X_t^{x,\pi^*} - Y_t$. Therefore substituting the explicit expression of the optimal strategy in (3.27), we obtain:

$$\begin{aligned} dJ_t^{\psi}(\pi) &= dM_t^{\pi^*} + \Theta_{t^-}\left[-\frac{(b_t K_t + c_t)^2}{a_t} + 2K_t(g_t^2 - u_t) - K_t^2 g_t^1 + v_t\right]dt - g_t^3 dt \\ &= dM_t^{\pi^*} + \Theta_{t^-}\left[-K_t^2\left(g_t^1 + \frac{b_t^2}{a_t}\right) + 2K_t\left(g_t^2 - u_t - \frac{b_t c_t}{a_t}\right)\right]dt + \left((v_t - \frac{c_t^2}{a_t})\Theta_{t^-} - g_t^3\right)dt. \end{aligned}$$

Then setting $g_t^1 + \frac{b_t^2}{a_t} = 0$, $g_t^2 - u_t - \frac{b_t c_t}{a_t} = 0$ and $(v_t - \frac{c_t^2}{a_t})\Theta_{t^-} - g_t^3 = 0$, we find that our coefficients g^1, g^2 and g^3 are given by:

$$g_t^1(\Theta_t, \theta_t^A, \theta_t^B, \beta_t) = -\frac{\left[\mu_t + \sum_{i\in\{A,B\}} \theta_t^i \sigma_t^i \lambda_t^i + \sigma_t \beta_t\right]^2}{\sigma_t^2 + \sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2 \lambda_t^i},$$

$$\begin{aligned} g_t^2(Y_t, U_t^A, U_t^B, Z_t) = &-\frac{\left[\mu_t + \sum_{i\in\{A,B\}} \theta_t^i \sigma_t^i \lambda_t^i + \sigma_t \beta_t\right]\left[\sigma_t Z_t + \sum_{i\in\{A,B\}}(1+\theta_t^i)\sigma_t^i U_t^i \lambda_t^i\right]}{\sigma_t^2 + \sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2 \lambda_t^i}, \\ &+ \sum_{i\in\{A,B\}} \theta_t^i U_t^i \lambda_t^i + \beta_t Z_t \end{aligned}$$

$$g_t^3(\xi_t, \epsilon_t^A, \epsilon_t^B, R_t) = \Theta_{t^-}\left[Z_t^2 + \sum_{i\in\{A,B\}} (U_t^i)^2(1+\theta_t^i)\lambda_t^i - \frac{\left(Z_t\sigma_t + \sum_{i\in\{A,B\}} \sigma_t^i U_t^i (1+\theta_t^i)\lambda_t^i\right)^2}{\sigma_t^2 + \sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2\lambda_t^i}\right].$$

Moreover the solution of the optimization problem 3.20 follows the quadratic form:

$$V(x) = \Theta_0(x - Y_0)^2 + \xi_0$$

Remark 3.5. *(Existence of the third BSDE)*

1. *If we find the solution of the first BSDE* $(\Theta, \theta^A, \theta^B, \beta) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$, *with the constraint* $\Theta \ge \delta > 0$ *and the second BSDE* $(Y, U^A, U^B, Z) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$ *then the solution of the third is given by:*

$$\xi_t = \mathbb{E}\left[\int_t^T \left(\left(v_s - \frac{c_s^2}{a_s}\right)\Theta_s\right) ds \middle| \mathcal{G}_t\right], \quad t \le T.$$

Then $|\xi| \in \mathcal{S}^\infty$ *and from representation Theorem 1.1, we deduce that the martingale part* M *of* ξ*:*

$$M_t = \int_0^t \sum_{i\in\{A,B\}} \epsilon^i_s dM^i_s + \int_0^t R_s dW_s$$

is BMO. *Moreover from Lemma 3.1,* ϵ^A *and* ϵ^B *are bounded. Therefore* $(\xi, \epsilon^A, \epsilon^B, R) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$

2. *In the complete market case, we have that the tracking error* $\xi \equiv 0$ *since the hedging is perfect.*

Now we give the Theorem which prove the existence of the solution of the first quadratic BSDE.

Theorem 3.4. *There exists a vector* $(\Theta, \theta^A, \theta^B, \beta) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$ *solution of the quadratic BSDE*

$$\frac{d\Theta_t}{\Theta_{t^-}} = -g^1_t(\Theta_t, \theta^A_t, \theta^B_t, \beta_t)dt + \theta^A_t dM^A_t + \theta^B_t dM^B_t + \beta_t dW_t, \qquad \Theta_T = 1.$$

Moreover there exists a non negative constant $\delta > 0$ *such that* $\Theta_t \geq \delta$ *for all* $t \in [0,T]$*. Given* $(\Theta, \theta^A, \theta^B, \beta)$*, we can prove the existence of solutions of* $(Y, U^A, U^B, Z) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$ *associated to* (g^2, ψ) *and* $(\xi, \epsilon^A, \epsilon^B, R) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$ *associated to the BSDE*$(g^3, 0)$*. Moreover, given this triplet solution* (Θ, Y, ξ) *of our system of BSDEs (3.26), the solution of the our optimization problem 3.20 is given by:*

$$V(x) = \Theta_0(x - Y_0)^2 + \xi_0$$

The proof of this Theorem will be given in the sequel in section 3.4.

3.2 Verification Theorem

Given the solution of the triple BSDEs in their respective spaces, we need to verify that the assertions defined in Theorem 3.3 hold true, i.e. the submartingale and martingale properties of the cost functional J is true and the strategy π^* defined in (3.29) is admissible. Moreover, we prove that the wealth process associated to π^* exists (satisfies a stochastic differential equation (SDE)).

We begin by proving the existence of the solution of the SDE for the wealth process associated to π^*.

Proposition 3.4. *Let* π^* *be the strategy, given by (3.29), then there exists a solution of the following SDE:*

$$dX^{x,\pi^*}_t = \pi^*_t \left[\mu_t dt + \sigma^A_t dM^A_t + \sigma^B_t dM^B_t + \sigma_t dW_t\right] \quad with \quad X^{x,\pi^*}_t = x. \tag{3.30}$$

Moreover, π^* *is admissible (i.e.* $\pi^* \in \mathcal{A}$*).*

Proof. The proof is divided in three steps. Firstly, we prove the existence of the SDE satisfies by the wealth associated to π^*, secondly we prove the squared integrability of this wealth at the horizon time T and thirdly we prove the admissibility of the strategy π^*.

The existence of the solution of the SDE for the wealth process: Plotting the expression of π^* given by (3.29) in (3.30) gives

$$dX_t^{x,\pi^*} = (\bar{b}_t X_t^{x,\pi^*} + \bar{c}_t)dt + (\bar{d}_t^A X_t^{x,\pi^*} + \bar{e}_t^A)dM_t^A + (\bar{d}_t^B X_t^{x,\pi^*} + \bar{e}_t^B)dM_t^B + (\bar{d}_t X_t^{x,\pi^*} + \bar{e}_t)dW_t \tag{3.31}$$

where the bounded processes are given by

$$\bar{b}_t = -\frac{b_t}{a_t}\mu_t, \quad \bar{c}_t = (b_t\frac{Y_t}{a_t} + c_t)\mu_t, \quad \bar{d}_t = -\frac{b_t}{a_t}\sigma_t$$
$$\bar{e}_t = (b_t\frac{Y_t}{a_t} + c_t)\sigma_t, \quad \bar{d}_t^i = -\frac{\bar{b}_t}{a_t}\sigma_t^i, \quad \bar{e}_t^i = (b_t\frac{Y_t}{a_t} + c_t)\sigma_t^i.$$

and processes a, b c are defined in (3.28). We recall, now, that the solution of the SDE:

$$d\phi_t = \phi_{t^-}\left[\bar{b}_t dt + \bar{d}_t^A dM_t^A + \bar{d}_t^B dM_t^B + \bar{d}_t dW_t\right] \quad \text{with} \quad \phi_0 = x$$

is given explicitely by

$$\phi_t = x\exp\left(\int_0^t \left(\bar{b}_s - \frac{1}{2}\bar{d}_s^2 - \sum_{i\in\{A,B\}} d_s^i\lambda_s^i\right) ds + \int_0^t \bar{d}_s dW_s\right) \prod_{i\in\{A,B\}} (1 + d_t^i H_t^i).$$

Therefore setting $X_t^{x,\pi^*} := L_t\phi_t$ with

$$dL_t := q_t dt + l_t^A dM_t^A + l_t^B dM_t^B + l_t dW_t, \quad L_0 = 1.$$

we find by integration by part formula that $dX_t^{x,\pi^*} = \phi_{t^-}dL_t + L_{t^-}d\phi_t + d[\phi, L]_t$. Hence,

$$dX_t^{x,\pi^*} = X_t^{x,\pi^*}\left[\bar{b}_t dt + \bar{d}_t^A dM_t^A + \bar{d}_t^B dM_t^B + \bar{d}_t dW_t\right] + \phi_{t^-}\left[q_t - \sum_{i\in\{A,B\}} d_t^i l_t^i \lambda_t^i\right] dt$$
$$+ \sum_{i\in\{A,B\}} \phi_{t^-} l_t^i (1 + d_t^i) dM_t^i + \phi_{t^-} l_t dW_t + l_t\phi_{t^-}\bar{d}_t dt.$$

Therefore from equation (3.31), we find, for $i \in \{A, B\}$, that $\bar{e}_t^i = \phi_{t^-} l_t^i (1 + d_t^i)$, $\bar{e}_t = \phi_{t^-} l_t$ and $\bar{c}_t = \phi_{t^-}\left(q_t - \sum_{i\in\{A,B\}} d_t^i l_t^i \lambda_t^i\right)$. We deduce that the process L is defined by:

$$L_t = 1 + \int_0^t \frac{1}{\phi_{s^-}}\left[\bar{c}_s + \sum_{i\in\{A,B\}} \frac{d_s^i e_s^i}{(1 + d_s^i)}\lambda_s^i\right] ds + \int_0^t \frac{\bar{e}_s}{\phi_{s^-}} dW_s + \int_0^t \sum_{i\in\{A,B\}} \frac{1}{\phi_{s^-}} \frac{e_s^i}{(1 + d_s^i)} dM_s^i$$

and $X_t^{x,\pi^*} = \phi_t L_t$ is a solution of the SDE (3.30).

Squared integrability of the strategy π^*: Let prove first that $X^{x,\pi^*} \in \mathcal{H}^2[0, T]$ and $X_T^{x,\pi^*} \in \mathrm{L}^2(\Omega, \mathcal{G}_T)$. We recall that $J_t^\psi(\pi^*) = \Theta_t(X_t^{x,\pi^*} - Y_t)^2 + \xi_t$ is a local martingale. Therefore there exists a sequence of localizing times $(T_i)_{i\in\mathbb{N}}$ for J_t^ψ such that for $t \le s \le T$

$$\mathbb{E}\left[J_{t\wedge T_i}^\psi(\pi^*)\right] = \Theta_0(x - Y_0)^2 + \xi_0.$$

From Remark 3.5, we have:

$$\mathbb{E}\left[\xi_{t\wedge T_i}-\xi_0\right]=-\mathbb{E}\left[\int_0^{t\wedge T_i}\left(v_s-\frac{c_s^2}{a_s}\right)\Theta_s ds\right],\quad t\le T.$$

where v, c and a are defined in Proposition 3.26. Since $a>0$, we have:

$$\mathbb{E}\left[\Theta_{t\wedge T_i}(X^{x,\pi^*}_{t\wedge T_i}-Y_{t\wedge T_i})^2\right]\le\Theta_0(x-Y_0)^2+\mathbb{E}\left[\int_0^{t\wedge T_i}v_s\Theta_s ds\right].$$

Moreover, since there exists a constant $\delta>0$ such that $\Theta_t>\delta$ and the process v is non negative, we can apply Fatou lemma and we find when i goes to infinity that

$$\delta\mathbb{E}\left[(X^{x,\pi^*}_t-Y_t)^2\right]\le\mathbb{E}\left[\Theta_t(X^{x,\pi^*}_t-Y_t)^2\right]\le\Theta_0(x-Y_0)^2+\mathbb{E}\left[\int_0^t v_s\Theta_s ds\right]$$

Therefore Z is BMO and the process $\theta^i,U^i\in\mathcal{S}^\infty[0,T]$ for $i=\{A,B\}$, we conclude $v\in\mathcal{H}^2[0,T]$. Hence, we have:

$$X^{x,\pi^*}-Y\in\mathcal{H}^2[0,T]$$
$$X^{x,\pi^*}_T-Y_T\in\mathrm{L}^2(\Omega,\mathcal{G}_T)$$

Since $Y\in\mathcal{S}^\infty[0,T]$, then we get the expected results: $X^{x,\pi^*}\in\mathcal{H}^2[0,T]$ and $X^{x,\pi^*}_T\in\mathrm{L}^2(\Omega,\mathcal{G}_T)$.

Admissibility of the strategy π^*: Let now prove that the strategy $\pi^*\in\mathcal{H}^2[0,T]$. Applying Itô's formula to $(X^{x,\pi^*})^2$, we get $d(X^{x,\pi^*}_t)^2=2X^{x,\pi^*}_{t^-}dX^{x,\pi}_t+d[X^{x,\pi^*}]_t$, then there exists a sequence of localizing times $(T_i)_{i\in\mathbb{N}}$ such that for all $t\le s\le T$:

$$x^2+\mathbb{E}\left[\int_0^{T\wedge T^i}|\pi^*_s|^2[\sigma_s^2+(\sigma^A)^2\lambda^A_s+(\sigma^B)^2\lambda^B_s]ds\right]\le\mathbb{E}\left[(X^{x,\pi^*}_{T\wedge T_i})^2\right]-2\mathbb{E}\left[\int_0^{T\wedge T_i}\pi^*_s\mu_s X^{x,\pi^*}_s ds\right]\tag{3.32}$$

Setting $K^\sigma_s=\sigma_s^2+(\sigma^A)^2\lambda^A_s+(\sigma^B)^2\lambda^B_s$ (K^σ is the so called the mean variance trade-off process), since the processes σ, σ^i, λ^i are bounded, there exists a constant K such that $K^\sigma\ge K$. Then we obtain

$$-2\pi^*_s\mu_s X^{x,\pi^*}_s\le\frac{2}{K}|X^{x,\pi^*}_s|^2|\mu_s|^2+\frac{K}{2}|\pi^*_s|^2,\quad 0\le s\le T$$

Therefore, combining this inequality with (3.32) gives

$$x^2+\mathbb{E}\left[\int_0^{T\wedge T^i}|\pi^*_s|^2K^\sigma_s ds\right]\le\mathbb{E}\left[(X^{x,\pi^*}_{T\wedge T_i})^2\right]+\mathbb{E}\left[\int_0^{T\wedge T_i}\frac{2}{K}|X^{x,\pi^*}_s|^2|\mu_s|^2ds\right]+\frac{K}{2}\mathbb{E}\left[\int_0^{T\wedge T_i}|\pi^*_s|^2ds\right].$$

Applying Fatou's lemma, when i goes to infinity, we get:

$$x^2+\mathbb{E}\left[\int_0^T|\pi^*_s|^2K^\sigma_s ds\right]\le\mathbb{E}\left[(X^{x,\pi^*}_T)^2\right]+\mathbb{E}\left[\int_0^T\frac{2}{K}|X^{x,\pi^*}_s|^2|\mu_s|^2ds\right]+\frac{K}{2}\mathbb{E}\left[\int_0^T|\pi^*_s|^2ds\right]$$

Therefore since $K^\sigma \geq K$, we finally obtain:

$$\frac{K}{2}\mathbb{E}\left[\int_0^T |\pi_s^*|^2 ds\right] \leq \mathbb{E}\left[\left(X_T^{x,\pi^*}\right)^2 - x^2\right] + \mathbb{E}\left[\int_0^T \frac{2}{K}|X_s^{x,\pi^*}|^2|\mu_s|^2 ds\right]$$

Since μ is bounded, $X^{x,\pi^*} \in \mathcal{H}^2[0,T]$ and $X_T^{x,\pi^*} \in \mathrm{L}^2(\Omega, \mathcal{G}_T)$, we conclude $\pi^* \in \mathcal{H}^2[0,T]$, so π^* is admissible. Note that this condition implies that $X^{x,\pi^*} \in \mathcal{S}^2[0,T]$ since all the asset's coefficients are bounded.

□

We now prove the submartingale and the martingale properties of the cost functional.

Proposition 3.5. *For any $\pi \in \mathcal{A}$, the process $J^\psi(\pi)$ is a true submartingale and a martingale for the strategy π^* given by (3.29). Moreover the strategy π^* is optimal for the minimization problem 3.20.*

Proof. Firstly, we prove the submartingale and the martingale property of the cost functional then secondly we prove that the strategy π^* is optimal.

First step: Let recall that for any $\pi \in \mathcal{A}$, the process $J^\psi(\pi)$ is a local submartingale and for π^*, $J^\psi(\pi^*)$ is a local martingale. Therefore, there exists a localizing increasing sequence of stopping times $(T_i)_{i\in\mathbb{N}}$ for J^ψ such that for $t \leq s \leq T$:

$$J^\psi_{t\wedge T_i}(\pi) \leq \mathbb{E}\left[J_{s\wedge T_i}(\pi)|\mathcal{G}_t\right] \quad \text{and } J^\psi_{t\wedge T_i}(\pi^*) = \mathbb{E}\left[J_{s\wedge T_i}(\pi^*)|\mathcal{G}_t\right] \quad \text{for any} \quad \pi \in \mathcal{A}. \tag{3.33}$$

Moreover, for any $\pi \in \mathcal{A}$, $J_t^\psi(\pi) = \Theta_t(X_t^{x,\pi} - Y_t) + \xi_t$ where Θ , Y and ξ are uniformly bounded and $X^{x,\pi} \in \mathcal{S}^2[0,T]$. Hence, taking the limit in (3.33) when i goes to infinity and applying dominated convergence Theorem, allow us to conclude.

Second step: For any $\pi \in \mathcal{A}$, we have from the submartingale property of $J^\psi(\pi)$ and the martingale property of $J^\psi(\pi^*)$:

$$\mathbb{E}\left[J_T^\psi(\pi)\right] \leq J_0^\psi(\pi) = \Theta_0(x - Y_0)^2 + \xi_0 = \mathbb{E}\left[J_T^\psi(\pi^*)\right]$$

we conclude π^* is the optimal strategy for the minimization problem 3.20.

□

3.3 Characterization of the VOM using BSDEs

Theorem 3.4 leads us to construct the VOM in some complete and incomplete markets. We will find also the price of the defaultable contingent claim ψ via the VOM. We will consider three different cases:

i. Complete market (where we assume $\mathbb{G} = \mathbb{F}$ and $\mathbb{G} = \mathbb{H}^A$)

ii. Incomplete market (where we consider only the case $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$).

iii. Incomplete market (where we consider the case $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$).

Remark 3.6. – *The case iii. corresponds to the more general case where the model depends on the market information (i.e. the filtration $\mathbb{F}$) and the defaults informations of the firms A and B. Indeed, in this set up, the model will depend to the default time of both firms. An economic interpretation of this case is a market with two firms where A is the main firm and B its insurance company. Then, the main firm A could make default and cause a default of its insurance. Then it is what we call a counterparty risk.*

– *The case ii. corresponds to a particular case where our model depends only to the default time of the firm A. In fact, in this set up, the model depends on the market information and the possible default of the firm A. It's can be view as a particular case of iii. with condition $\tau^B = \infty$ (i.e. no possible default of firm B).*

– *The first case in i., if $\mathbb{G} = \mathbb{F}$, corresponds to a model which depends only on the market information and not to the possible default of firms A and B. In a economic point of view, it is a simple model without default. In the second case, $\mathbb{G} = \mathbb{H}^A$, the model depends only to the possible default of the firm A and non more to the information given by the market.*

Remark 3.7. *We have explicit solution of the VOM with respect to the process Θ in the first two cases.*

3.3.1 Complete market

If we assume that $\mathbb{G} = \mathbb{F}$ (we do not consider the default impact of firms A and B on the asset dynamics of the firm A) or $\mathbb{G} = \mathbb{H}^A$ (we don't consider the market noise) then our financial market is complete. Hence, the VOM is the unique risk neutral probability and its dynamics can be found explicitly. Our goal in this part is so to find the solution of the triple BSDEs given the VOM $\bar{\mathbb{P}}$.

Proposition 3.6. *Let $\bar{\mathbb{P}}$ be the VOM (the unique risk neutral probability) and let define $\bar{Z}_T$ be the Radon Nikodym density of $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$. We denote $\bar{Z}_t = \mathbb{E}\left[\bar{Z}_T|\mathcal{G}_t\right]$, then for all $t \leq T$, we have that*

$$\Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}$$

Moreover, for all $t \in [0, T]$, we have that $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$ and $\xi_t \equiv 0$.

Proof. We will consider the two cases $\mathbb{G} = \mathbb{F}$ and $\mathbb{G} = \mathbb{H}^A$.

First case: Let consider the case where $\mathbb{G}$ is equal to $\mathbb{F}$ and let the process L defined by the stochastic differential equation given by

$$dL_t = L_{t^-}\rho_t dW_t$$

where $\rho W \in \text{BMO}$, using Itô's formula we find:

$$\begin{aligned} d\left(\frac{L_t^2}{\Theta_t}\right) &= \frac{L_t^2}{\Theta_t}\left[(2\rho_t - \beta_t)dW_t + (\beta_t^2 + g_t^1 - 2\beta_t\rho_t + \rho_t^2)dt\right] \\ &= \frac{L_t^2}{\Theta_t}\left[(2\rho_t - \beta_t)dW_t + \left((\beta_t - \rho_t)^2 - (\frac{\mu_t}{\sigma_t} + \beta_t)^2\right)dt\right] \\ &= \frac{L_t^2}{\Theta_t}\left[(2\rho_t - \beta_t)dW_t + \left((-\rho_t - \frac{\mu_t}{\sigma_t})(2\beta_t + \rho_t + \frac{\mu_t}{\sigma_t})\right)dt\right] \end{aligned}$$

Then if we set, for all $t \leq T$, that $\rho_t := -\frac{\mu_t}{\sigma_t}$ and using the bound condition of $\left(\frac{1}{\Theta}, \mu, \sigma\right)$ and the BMO property of β, we obtain that the process $\frac{L^2}{\Theta}$ is a true martingale. Therefore we get:

$$\mathbb{E}\left(\frac{L_T^2}{\Theta_T}\Big|\mathcal{G}_t\right) = \frac{L_t^2}{\Theta_t}, \quad t \leq T$$

Since $\Theta_T = 1$, we find the expected result. Moreover we obtain that $L = \bar{Z}$ which is the Radon-Nikodym of the unique risk neutral probability and $g_t^2 = -\frac{\mu_t}{\sigma_t}Z_t$, $g_t^3 = 0$, then $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$ and $\xi_t = 0, t \leq T$.

Second case: Let now consider the case where $\mathbb{G}$ is equal to $\mathbb{H}$ and let the process L define by the stochastic differential equation given by

$$dL_t = L_{t^-}\rho_t^A dM_t^A$$

where $\rho^A M^A \in \text{BMO}$, using Itô's formula we find:

$$\begin{aligned} d\left(\frac{L_t^2}{\Theta_t}\right) &= \frac{L_{t^-}^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right)dM_t^A + \left(\frac{((\theta_t^A)^2 + (\rho_t^A)^2 - 2\rho_t^A\theta_t^A)\lambda_t^A}{1+\theta_t^A} + g_t^1\right)dt\right] \\ &= \frac{L_{t^-}^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right)dM_t^A + \frac{1}{1+\theta_t^A}\left((\rho_t^A - \theta_t^A)^2 - (\frac{\mu_t}{\sigma_t^A\lambda_t^A} + \theta_t^A)^2\right)\lambda_t^A dt\right] \\ &= \frac{L_{t^-}^2}{\Theta_{t^-}}\left[\left(\frac{(1+\rho_t^A)^2}{1+\theta_t^A} - 1\right)dM_t^A + \frac{1}{1+\theta_t^A}\left((\rho_t^A + \frac{\mu_t}{\sigma_t^A\lambda_t^A})(-2\theta_t^A + \rho_t^A - \frac{\mu_t}{\sigma_t^A\lambda_t^A})\right)\lambda_t^A dt\right] \end{aligned}$$

then if we set for all $t \leq T$

$$\rho_t^A := -\frac{\mu_t}{\lambda_t^A\sigma_t^A}$$

then using the bound condition of $\Theta, \mu, \sigma^A, \theta^A$, the process $\frac{L^2}{\Theta}$ is a true martingale. Hence we get:

$$\mathbb{E}\left(\frac{L_T^2}{\Theta_T}\Big|\mathcal{G}_t\right) = \frac{L_t^2}{\Theta_t}, \quad t \leq T$$

Since $\Theta_T = 1$, we find again the expected result. Moreover $L = \bar{Z}$ the Radon-Nikodym of the unique risk neutral probability and $g_t^2 = -\frac{\mu_t}{\lambda_t^A}U_t^A$, $g_t^3 = 0$, then $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$ and $\xi_t = 0, t \leq T$.

□

Remark 3.8. *We have proved that we can find the existence of solution of the triple BSDEs using only the explicitly given VOM.*

3.3.2 Incomplete market

In the incomplete market case, the remark 3.8 doesn't hold true. The VOM depends on the dynamics of $(\Theta, \theta^A, \theta^B, \beta)$. In the particular case where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$, we can find that the Proposition 3.6 holds true. But in the more general case $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$, we can not prove the existence of the VOM but we still characterize the process Θ with some martingale measure.

Proposition 3.7. *Let consider the incomplete market $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$, then the VOM $\bar{\mathbb{P}}$ defines the local martingale measure $\mathbb{Q}$ which minimizes the L^2-norm of $Z^{\mathbb{Q}}$, $\bar{Z}_T$ represents the Radon Nikodym density of $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ and $\bar{Z}_t = \mathbb{E}\left[\bar{Z}_T|\mathcal{G}_t\right]$. We find, for all $t \le T$,*

$$\Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}.$$

Moreover

$$Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right].$$

In the more general case, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$, we can only prove that there exists a martingale measure $\bar{\mathbb{P}}$ such that for all $t \le T$:

$$\Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]} \quad \textit{and} \quad Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$$

Proof. **First step:** Consider the case where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$ and $\mathbb{Q}$ a martingale measure for the asset D^A. Let define $Z_T^{\mathbb{Q}}$ its Radon Nikodym density with respect to $\mathbb{P}$ on $\mathcal{G}_T$. We define the process $Z_t^{\mathbb{Q}} = \mathbb{E}\left[Z_T^{\mathbb{Q}}|\mathcal{G}_t\right]$. Using martingale representation Theorem 1.1, there exists two $\mathbb{G}$-predictable processes ρ^A and ρ such that

$$dZ_t^{\mathbb{Q}} = Z_{t^-}^{\mathbb{Q}}\left[\rho_t^A dM_t^A + \rho_t dW_t\right]$$

Using Itô's formula, we find:

$$d\left(\frac{\left(Z_t^{\mathbb{Q}}\right)^2}{\Theta_t}\right) = \frac{\left(Z_{t^-}^{\mathbb{Q}}\right)^2}{\Theta_{t^-}}\left[\left(\frac{\left(1+\rho_t^A\right)^2}{1+\theta_t^A} - 1\right)dM_t^A + (2\rho_t - \beta_t)dW_t + j_t dt\right] \tag{3.34}$$

where $j_t = (\rho_t - \beta_t)^2 + \frac{\left(\rho_t^A - \theta_t^A\right)^2}{1+\theta_t^A}\lambda_t^A + g_t^1$. Since $\mathbb{Q}$ is a martingale measure for D^A we get using (2.13) that

$$\mu_t^A + \rho_t^A \sigma_t^A \lambda_t^A + \rho_t \sigma_t = 0$$

Hence using this equation we can find ρ^A using ρ and plotting this result on the expression of j. We obtain

$$j_t = (\rho_t - \beta_t)^2 + \frac{\left(\mu_t + \sigma_t\rho_t + \sigma_t^A\theta_t^A\lambda_t^A\right)^2}{(1+\theta_t^A)\left(\sigma_t^A\right)^2\lambda_t^A} - \frac{\left(\mu_t + \beta_t\sigma_t + \theta_t^A\sigma_t^A\lambda_t^A\right)^2}{(1+\theta_t^A)\left(\sigma_t^A\right)^2\lambda_t^A + \sigma_t^2}$$

Let now define

$$\bar{\rho}_t = \rho_t - \beta_t, \quad \bar{a}_t = \sigma_t^2 + (1+\theta_t^A)\left(\sigma_t^A\right)^2\lambda_t^A \quad \textit{and} \quad \bar{b}_t = \mu_t + \sigma_t\beta_t + \sigma_t^A\theta_t^A\lambda_t^A$$

then we get:

$$j_t = \frac{1}{(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A}\left[\bar{a}_t\bar{\rho}_t + 2\bar{\rho}_t\bar{b}_t\sigma_t + \frac{\bar{b}_t^2\sigma_t^2}{\bar{a}_t}\right] = \frac{\bar{a}_t}{(1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A}\left(\bar{\rho}_t + \frac{\bar{b}_t\sigma_t}{\bar{a}_t}\right)^2 > 0.$$

(3.34), $j \geq 0$ and the fact that the process $\frac{(Z^{\mathbb{Q}})^2}{\Theta}$ is a submartingale (since $Z^{\mathbb{Q}}$ is a martingale and $\frac{1}{\Theta} \in \mathcal{S}^\infty[0,T]$), we deduce $\mathbb{E}\left[\frac{(Z_T^{\mathbb{Q}})^2}{\Theta_T}\right] \geq \frac{(Z_0^{\mathbb{Q}})^2}{\Theta_0}$, since $\Theta_T = 1$ and $Z_0^{\mathbb{Q}} = 1$. Finally we get for any martingale measure for D^A that $\mathbb{E}\left[(Z_T^{\mathbb{Q}})^2\right] \geq \frac{1}{\Theta_0}$. Moreover, if we set $\bar{\rho}_t = -\frac{\bar{b}_t\sigma_t}{\bar{a}_t}$, then $\bar{Z}$ is a true martingale measure since $(\Theta, \theta^A, \theta^B\beta) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \text{BMO}$ and μ, σ^A, σ^B are bounded (the process b,a, ρ and ρ^A are bounded). We call $\bar{\mathbb{P}}$ the martingale measure under this condition then $\mathbb{E}\left[\bar{Z}_T^2\right] = \frac{1}{\Theta_0}$. We deduce $\bar{\mathbb{P}}$ is the martingale measure which minimizes the L^2-norm of Z and $\bar{\Theta}_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}, t \leq T$. Using the explicit expression of ρ we find:

$$\rho_t = -\frac{\sigma_t\bar{b}_t}{\bar{a}_t} + \beta_t \quad \text{and} \quad \rho_t^A = -\frac{(1+\theta_t^A)\sigma_t^A\bar{b}_t}{\bar{a}_t} + \theta_t^A$$

Moreover since

$$\begin{aligned} g_t^2 &= \frac{-\bar{b}_t(\sigma_t Z_t + (1+\theta_t^A)U_t^A\sigma_t^A\lambda_t^A)}{\bar{a}_t} + \beta_t Z_t + U_t^A\lambda_t^A \\ &= Z_t\left(-\frac{\bar{b}_t\sigma_t}{\bar{a}_t} + \beta_t\right) + U_t^A\left(-\frac{(1+\theta_t^A)\sigma_t^A\bar{b}_t}{\bar{a}_t} + \theta_t^A\right)\lambda_t^A \\ &= Z_t\rho_t + U_t^A\rho_t^A\lambda_t^A \end{aligned}$$

then we conclude that $Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right]$. Therefore the characterization of the price of ψ (using Mean-Variance approach) and the VOM in this incomplete case is well defined using $(\Theta, \theta^A, \theta^B, \beta)$ associated to the first BSDE.

Second step: We consider now the more general case where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$. Let consider $\mathbb{Q}$ a martingale measure for the asset D^A and let define $Z_T^{\mathbb{Q}}$ its Radon Nikodym density with respect to $\mathbb{P}$ on $\mathcal{G}_T$. We can define the process $Z_t^{\mathbb{Q}} = \mathbb{E}\left[Z_T^{\mathbb{Q}}|\mathcal{G}_t\right]$. Using martingale theorem representation 1.1 there exists $\mathbb{G}$-predictable processes ρ^A, ρ^B and ρ such that

$$dZ_t^{\mathbb{Q}} = Z_{t-}^{\mathbb{Q}}\left[\rho_t^A dM_t^A + \rho_t^B dM_t^B + \rho_t dW_t\right]$$

Using Itô's formula, we find:

$$d\left(\frac{(Z_t^{\mathbb{Q}})^2}{\Theta_t}\right) = \frac{(Z_{t-}^{\mathbb{Q}})^2}{\Theta_{t-}}\left[\sum_{i\in\{A,B\}}\left(\frac{(1+\rho_t^i)^2}{1+\theta_t^i} - 1\right)dM_t^i + (2\rho_t - \beta_t)dW_t + j_t dt\right]$$

where $j_t = (\rho_t - \beta_t)^2 + \sum_{i\in\{A,B\}}\frac{(\rho_t^i - \theta_t^i)^2}{1+\theta_t^i}\lambda_t^i + g_t^1$. Since $\mathbb{Q}$ is a martingale measure for D^A we get by (2.13)

$$\mu_t^A + \sum_{i\in\{A,B\}}\rho_t^i\sigma_t^i\lambda_t^i + \rho_t\sigma_t = 0$$

Hence using this equation, we can find ρ^A using ρ and ρ^B and then plotting this result on the expression of j. Let first recall a notation:

$$a_t = \sigma_t^2 + \sum_{i\in\{A,B\}} (1+\theta_t^i)(\sigma_t^i)^2\lambda_t^i \quad and \quad b_t = \mu_t + \sigma_t\beta_t + \sum_{i\in\{A,B\}} \theta_t^i\sigma_t^i\lambda_t^i$$

so we find:

$$\begin{aligned}
C_t :=& (1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A j_t \\
&= (\rho_t-\beta_t)^2[\sigma_t^2 + (1+\theta_t^A)(\sigma_t^A)^2\lambda_t^A] + \frac{(\rho_t^B-\theta_t^B)^2}{1+\theta_t^B}\lambda_t^B \left[\sum_{i\in\{A,B\}}(1+\theta_t^i)(\sigma_t^i)^2\lambda_t^i\right] \\
&+ \frac{b_t^2}{a_t}\left[\sigma_t^2 + (1+\theta_t^B)(\sigma_t^B)^2\lambda_t^B\right] + 2(\rho_t^B-\theta_t^B)(\rho_t-\beta_t)\sigma_t\sigma_t^B\lambda_t^B \\
&+ 2b_t\left[(\rho_t-\beta_t)\sigma_t + (\rho_t^B-\theta_t^B)\sigma_t^B\lambda_t^B\right]
\end{aligned}$$

Then from the two first terms, we add and remove an additional process to find the process a, we get:

$$\begin{aligned}
C_t &= \left[(\rho_t-\beta_t)^2a_t + \frac{b_t^2}{a_t}\sigma_t^2 + 2b_t(\rho_t-\beta_t)\sigma_t\right] + (1+\theta_t^B)\lambda_t^B\Big[\frac{(\rho_t^B-\theta_t^B)^2}{(1+\theta_t^B)^2}a_t + 2b_t\sigma_t^B\frac{\rho_t^B-\theta_t^B}{1+\theta_t^B} \\
&\quad + \frac{b_t^2}{a_t^2}(\sigma_t^B)^2\Big] + (1+\theta_t^B)\lambda_t^B\left[2(\rho_t-\beta_t)\frac{(\rho_t^B-\theta_t^B)}{(1+\theta_t^B)}\sigma_t\sigma_t^B - (\rho_t-\beta_t)^2(\sigma_t^B)^2 - \frac{(\rho_t^B-\theta_t^B)^2}{(1+\theta_t^B)^2}\sigma_t^2\right]
\end{aligned}$$

Finally we find a more explicit expression of C:

$$\begin{aligned}
C_t &= a_t\left[\left((\rho_t-\beta_t) + \frac{b_t}{a_t}\sigma_t\right)^2 + (1+\theta_t^B)\lambda_t^B\left(\frac{(\rho_t^B-\theta_t^B)}{1+\theta_t^B} + \frac{b_t\sigma_t^B}{a_t}\right)^2\right] \\
&\quad -(1+\theta_t^B)\lambda_t^B(\sigma_t^B)^2\left((\rho_t-\beta_t) - \frac{\sigma_t}{\sigma_t^B}\frac{\rho_t^B-\theta_t^B}{1+\theta_t^B}\right)^2
\end{aligned}$$

It follows that if we set $\rho_t - \beta_t := -\frac{b_t}{a_t}\sigma_t$ and $\rho_t^B - \theta_t^B := -(1+\theta_t^B)\sigma_t^B\frac{b_t}{a_t}$, then we find $j = 0$ and $\rho_t^A - \theta_t^A = -\sigma_t^A\frac{b_t}{a_t}$. Since $(\frac{1}{\Theta}, \theta^A, \theta^B, \beta) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times$ BMO and μ, σ^A, σ^B and σ are bounded then the processes b, a, ρ, ρ^A and ρ^B are bounded too. Therefore, we deduce there exists a martingale measure $\bar{\mathbb{P}}$ such that

$$\delta \le \Theta_t = \frac{\bar{Z}_t^2}{\mathbb{E}\left[\bar{Z}_T^2|\mathcal{G}_t\right]}, \quad t \le T. \tag{3.35}$$

Moreover we find, for all $t \le T$, that

$$g_t^2 = Z_t\rho_t + \sum_{i\in\{A,B\}} U_t^i\rho_t^i\lambda_t^i$$

then

$$Y_t = \bar{\mathbb{E}}\left[\psi|\mathcal{G}_t\right].$$

□

Remark 3.9. *(About the VOM)*

– *To identify that* $\bar{\mathbb{P}}$ *is the VOM in the general case where* $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A \vee \mathbb{H}^B$, *we should prove that* $j \geq 0$ *as in the first case of the previous Proposition. But from the last expression of* j, *we can not prove that this condition holds true. However, we can remark that the assertion of VOM will be justify if one of the following equality is satisfied:*

$$\sigma_t^B(\rho_t - \beta_t) = \sigma_t \frac{\rho_t^B - \theta_t^B}{1+\theta_t^B}, \quad \sigma_t^A \frac{\rho_t^B - \theta_t^B}{1+\theta_t^B} = \sigma_t^B \frac{\rho_t^A - \theta_t^A}{1+\theta_t^A} \quad \text{or} \quad \sigma_t^A(\rho_t - \beta_t) = \sigma_t \frac{\rho_t^A - \theta_t^A}{1+\theta_t^A}.$$

– *The generalization of the expectation under a* σ*-measure (* $Y_t = \bar{\mathbb{E}}[\psi|\mathcal{G}_t]$*) was defined by Cerny and Kallsen in [3] p 1512. Moreover, given the solution of the first BSDE,* $(\Theta, \theta^A, \theta^B, \beta) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$, *with the constraint* $\Theta \geq \delta > 0$, *the martingale* $\bar{Z}$ *satisfies (3.35), since* ψ *is bounded, we conclude:*

$$|Y_t| \leq 2\mathbb{E}\left[\frac{\bar{Z}_T^2}{\bar{Z}_t^2}|\mathcal{G}_t\right] + 2\mathbb{E}\left[[\psi|^2|\mathcal{G}_t\right] \leq 2\left[\frac{1}{\delta} + ||\psi||_\infty^2\right]$$

Therefore $Y \in \mathcal{S}^\infty[0,T]$ *and from representation theorem 1.1, the martingale part* M *of* Y *given by*

$$M_t = \int_0^t \sum_{i\in\{A,B\}} U_s^i dM_s^i + \int_0^t Z_s dW_s$$

is BMO*. Moreover from Lemma 3.1 in Appendix, since* $Y \in \mathcal{S}^\infty[0,T]$ *we obtain that* θ^A *and* θ^B *are bounded. We conclude if the solution of the first BSDE exists* $(\Theta, \theta^A, \theta^B, \beta) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$, *with the constraint* $\Theta \geq \delta > 0$, *that the solution of the second BSDE* $(Y, U^A, U^B, Z) \in \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$ *exists.*

3.4 Proof of Theorem 3.4

We prove in this part the existence of $(\Theta, \theta^A, \theta^B, \beta)$ in the space $\mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathcal{S}^\infty[0,T] \times \mathrm{BMO}$, with the constraint $\Theta > \delta$. Moreover, we recall that given the solution of this first BSDE, the existence of the second and the third BSDEs is given in Remark 3.5 and 3.9.

Note that to prove the existence of $(\Theta, \theta^A, \theta^B, \beta)$, we do not need the assumption that the VOM exists and should satisfied the $\mathbb{R}_2(\mathbb{P})$ condition (this assumption implies that the Radon-Nikodym of the VOM $\bar{\mathbb{P}}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ is non-negative). Moreover, if $(\Theta, \theta^A, \theta^B, \beta)$ is defined such that $\bar{Z}$ is non negative implies that $\bar{\mathbb{P}}$ satisfies the $R_2(\mathbb{P})$ condition.

In fact, in general discontinuous filtration it is difficult to prove that we can find $(\Theta, \theta^A, \theta^B, \beta)$ solution of the first BSDE such that $\Theta > 0$ (see [23] for more discussions about the difficulty). Indeed in the set up of [23], the author make hypothesis that all the asset's coefficients are $\mathbb{F}$-predictable. This strong hypothesis makes the jump part of process Θ equals to zero. In our framework, this hypothesis can't be satisfied since the intensities processes are $\mathbb{G}$-adapted. Hence, we deal with splitting method of BSDE defined by [18] , to prove that even the jump part of the process is not equals to zero, we can split the jump BSDE in continuous BSDEs such that each BSDE have a solution in a good space. The proof is divided in two parts. In the first part, we will give the splitted BSDEs in this framework and in the second part we will solve recursively each BSDE.

First step (The splitting of the jump BSDE) Let define, for all $t \in [0,T]$, $\bar{g}_t = \Theta_{t^-} g^1_t$, $\bar{\theta}^i_t = \Theta_{t^-}\theta^i_t$ for $i \in \{A,B\}$, $\bar{\theta}_t = \bar{\theta}^A_t 1_{\{t<\tau^A\}} + \bar{\theta}^B_t 1_{\{\tau^A \le t \le \tau^B\}}$ and $\bar{\beta}_t = \Theta_{t^-}\beta_t$. Then we can define the BSDE $(\bar{g}, \Theta_T)$ given by:

$$d\Theta_t = -\bar{f}_t dt + \bar{\theta}^A_t dH^A_t + \bar{\theta}^B_t dH^B_t + \bar{\beta}_t dW_t \quad \text{with} \quad \Theta_T = 1.$$

where $\bar{f}_t = \bar{g}_t + \bar{\theta}^A_t \lambda^A_t + \bar{\theta}^B_t \lambda^B_t$. We define too

$$\Delta_k = \{(l_1, \cdots l_k) \in (\mathbb{R}^+)^k : l_1 \le \cdots \le l_k\}, \quad 1 \le k \le 2.$$

Since we work with the same assumption (density assumption) and notation as in [18], then we can decomposed Θ_T and $\bar{g}$ between each default events such that:

$$\Theta_T \;=\; \gamma^0 1_{\{\{0 \le T < \tau_A\}\}} + \gamma^1(\tau^A) 1_{\{\tau^A \le T \le \tau^B\}} + \gamma^2(\tau^A, \tau^B) 1_{\{\tau^B < T\}}$$

and

$$\begin{aligned} \bar{f}_t(\Theta_t, \bar{\theta}_t, \bar{\beta}_t) \;&=\; \bar{f}^0_t(\Theta_t, \bar{\theta}_t, \bar{\beta}_t) 1_{\{0 \le t < \tau_A\}} + \bar{f}^1_t(\Theta_t, \bar{\theta}_t, \bar{\beta}_t, \tau^A) 1_{\{\tau^A \le t \le \tau^B\}} && (3.36)\\ &+\; \bar{f}^2_t(\Theta_t, \bar{\theta}_t, \bar{\beta}_t, (\tau^A, \tau^B)) 1_{\{\tau^B < t\}} && (3.37)\end{aligned}$$

where γ^0 is $\mathcal{F}_T$-measurable, γ^k is $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k)$-measurable for $k = \{1,2\}$, $\bar{g}^0$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable and $\bar{g}^k$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Delta^k)$. Moreover, since $\Theta_T = 1$ (bounded) (see proposition 3.1 in Kharroubi and Lim [18]) we have that the variables $\gamma^k(l_{(k)}) = 1$, for $k = \{0,1,2\}$. Let now give the main result of splitting BSDE which is a first step to prove the existence of $(\Theta, \bar{\theta}^A, \bar{\theta}^B, \bar{\beta})$. Let $l_{(2)} = (l_1, l_2) \in \Delta_2$ and assume that the following BSDE:

$$d\Theta^2_t(l_{(2)}) = -\bar{f}^2_t\left(\Theta^2_t(l_{(2)}), 0, \bar{\beta}^2_t(l_{(2)}), l\right) dt + \bar{\beta}^2_t(l_{(2)}) dW_t, \quad \Theta^2_T(l_{(2)}) = \gamma^2(l_{(2)}). \tag{3.38}$$

admits a solution $\left(\Theta^2_t((l_{(k+1)})), \bar{\beta}^2_t((l_{(k+1)}))\right) \in S^\infty([l_2 \wedge T, T]) \times \mathcal{H}^2[l_2 \wedge T, T]$ and for $k = \{0,1\}$

$$\begin{aligned} d\Theta^k_t(l_{(k)}) \;&=\; -\bar{f}^k_t\left(\Theta^k_t(l_{(k)}), (\Theta^{k+1}_t(l_{(k)}, t) - \Theta^k_t(l_{(k)})), \bar{\beta}^k_t(l_{(k)}), l_{(k)}\right) dt + \bar{\beta}^k_t(l_{(k)}) dW_t,\\ \Theta^k_T(l_{(k)}) \;&=\; \gamma^k(l_{(k)}). \end{aligned} \tag{3.39}$$

admits a solution $\left(\Theta^k(l_{(k)}, \bar{\beta}^k(l_{(k)})\right) \in \mathcal{S}^\infty([l_k \wedge T, T]) \times \mathcal{H}^2[l_k \wedge T, T]$, where $l_{(k)} = (l_1, \cdots l_k)$. Then $(\Theta, \bar{\theta}^A, \bar{\theta}^B, \bar{\beta})$ is given by [18]:

$$\begin{aligned} \Theta_t &= \Theta^0_t 1_{\{t<\tau^A\}} + \Theta^1_t(\tau^A) 1_{\{\tau^A \le t \le \tau^B\}} + \Theta^2_t(\tau^A, \tau^B) 1_{\{\tau^B < t\}}\\ \bar{\beta}_t &= \bar{\beta}^0_t 1_{\{t<\tau^A\}} + \bar{\beta}^1_t(\tau^A) 1_{\{\tau^A \le t \le \tau^B\}} + \bar{\beta}^2_t(\tau^A, \tau^B) 1_{\{\tau^B < t\}}\\ \bar{\theta}^B_t &= \Theta^2_t(\tau_A, t) - \Theta^1_t(\tau_A)\\ \bar{\theta}^A_t &= \Theta^1_t(t) - \Theta^0_t(t) \end{aligned} \tag{3.40}$$

Therefore, to prove the existence of $(\Theta, \theta^A, \theta^B, \beta)$ we should prove the existence of the solution of the BSDEs (3.38) and (3.39).

Second step (the recursive approach) : We prove recursively the existence of the solution of BSDEs. First, we prove the existence of the solution of (3.38) and secondly assuming that the solution of (3.39) exists and satisfies the constraint for step $k+1$, we prove the same assertion for the step k.

1. Let consider the BSDE (3.38):

$$d\Theta_t^2(l_{(2)}) = -\bar{f}_t^2(\Theta_t^2(l_{(2)}), 0, \bar{\beta}_t^2(l_{(2)}), l_{(2)})dt + \bar{\beta}_t^2(l_{(2)})dW_t, \quad \Theta_T^2(l_{(2)}) = \gamma^2(l_{(2)}).$$

Since the coefficient $\bar{g}$ is given by:

$$\bar{g}_t(\Theta_t, \bar{\theta}, \beta_t) = -\frac{\left[\mu_t\Theta_t + \sum_{i\in\{A,B\}} \bar{\theta}_t^i \sigma_t^i \lambda_t^i + \sigma_t \bar{\beta}_t\right]^2}{\Theta_t\sigma_t^2 + \sum_{i\in\{A,B\}}(\Theta_t + \bar{\theta}_t^i)(\sigma_t^i)^2\lambda_t^i} \tag{3.41}$$

From the predictable decomposition of the composition of assets coefficient:

$$\begin{aligned}
\sigma_t &= \sigma_t^0 1_{\{t<\tau^A\}} + \sigma_t^1(\tau^A)1_{\{\tau^A\le t\le\tau^B\}} + \sigma_t^2(\tau^A,\tau^B)1_{\{\tau^B<t\}} \\
\mu_t &= \mu_t^0 1_{\{t<\tau^A\}} + \mu_t^1(\tau^A)1_{\{\tau^A\le t\le\tau^B\}} + \mu_t^2(\tau^A,\tau^B)1_{\{\tau^B<t\}} \\
\sigma_t^A &= \sigma_t^{1,0} 1_{\{t<\tau^A\}} + \sigma_t^{1,1}(\tau^A)1_{\{\tau^A\le t\le\tau^B\}} + \sigma_t^{1,2}(\tau^A,\tau^B)1_{\{\tau^B<t\}}, \\
\sigma_t^B &= \sigma_t^{2,0} 1_{\{t<\tau^A\}} + \sigma_t^{2,1}(\tau^A)1_{\{\tau^A\le t\le\tau^B\}} + \sigma_t^{2,2}(\tau^A,\tau^B)1_{\{\tau^B<t\}},
\end{aligned} \tag{3.42}$$

And by our model assumption $\tau^A < \tau^B$, we get

$$\lambda_t^A = \lambda_t^{1,0}1_{\{t<\tau^A\}} \quad \text{and} \quad \lambda_t^B = \lambda_t^{2,1}(\tau^A)1_{\{\tau^A\le t\le\tau^B\}}$$

We so find:

$$\bar{f}_t^2(\Theta_t^2(l_{(2)}), 0, \bar{\beta}_t^2(l_{(2)}), l_{(2)}) = \Theta_t^2(l_{(2)})\left[\frac{\mu_t^2(l_{(2)})}{(\sigma_t^2(l_{(2)}))^2} + \frac{\bar{\beta}_t^2(l_{(2)})}{\Theta_t^2(l_{(2)})}\right]^2, \quad t\in[0,T]$$

Using the result of Section 3.3, in the complete market when $\mathbb{G} = \mathbb{F}$, we conclude:

$$\Theta_t^2(l_{(2)}) = \frac{Z_t^2(l_{(2)})}{\mathbb{E}\left[\frac{Z_T^2(l_{(2)})}{\gamma^2(l_{(2)})}\right]}, \quad t\le T, l_{(2)}\in\Delta_2 \tag{3.43}$$

where the family of processes $Z(.)$ satisfies the SDE

$$\frac{dZ_t(l_{(2)})}{Z_t(l_{(2)})} = -\frac{\mu_t^2(l_{(2)})}{\sigma_t^2(l_{(2)})}dW_t$$

with $Z_0(l_{(2)}) = 1$. Since μ^2 and σ^2 are bounded, the martingale $M_t(l_{(2)}) := \int_0^t \frac{\mu_s^2(l_{(2)})}{\sigma_s^2(l_{(2)})}dW_s$ is BMO. We deduce so that $Z(l_{(2)})$ satisfies the $R_2(\mathbb{P})$ inequality. Moreover $\gamma^2(l_{(2)}) = 1$, we conclude that there exists a constant $\delta^2 > 0$ such that for all $t\in[0,T]$ and $l_{(2)}\in\Delta_2$, $\Theta_t^2(l_{(2)}) \ge \delta^2$. The existence of $\bar{\beta}^2(l_{(2)})$ is given by the martingale part of the process given by (3.43). Moreover since $\Theta^2(l_{(2)})$ is bounded then the coefficient $\bar{g}^2$ satisfies a quadratic growth with respect to $\bar{\beta}^2(l_{(2)})$. Therefore since the terminal condition $\gamma^2(l_{(2)})$ is bounded, we conclude from Kobylanski [19], that $\bar{\beta}(l_{(2)})$ is BMO.

2. Let assume now that there exists a solution which satisfies the constraint for the step $k+1$, that means the pair $(\Theta^{k+1}(l_{(k+1)}), \bar{\beta}_t^{k+1}(l_{(k+1)})) \in \mathcal{S}^\infty[l_{k+1}, T] \times \text{BMO}$ and that there exists a non negative constant δ^{k+1} such that $\Theta^{k+1}(l_{(k+1)}) \geq \delta^{k+1}$. Let now prove the existence of the pair $(\Theta^k(l_{(k)}), \bar{\beta}^k(l_{(k)})) \in \mathcal{S}^\infty[l_k, T] \times \text{BMO}$ at step k:

$$d\Theta_t^k(l_{(k)}) = -\bar{f}_t^k\left(\Theta_t^k(l_{(k)}), (\Theta_t^{k+1}(l_{(k)}, t) - \Theta_t^k(l_{(k)})), \bar{\beta}_t^k(l_{(k)}), l_{(k)}\right)dt + \bar{\beta}_t^k(l_{(k)})dW_t,$$
$$\Theta_T^k(l_{(k)}) = \gamma^k(l_{(k)}).$$

From the decomposition of (3.41), we find:

$$\begin{aligned}
&\bar{f}_t^k\left(\Theta_t^k(l_{(k)}), (\Theta_t^{k+1}(l_{(k)}, t) - \Theta_t^k(l_{(k)})), \bar{\beta}_t^k(l_{(k)}), l_{(k)}\right) \\
&= -\frac{\left[\mu_t^k(l_{(k)})\Theta_t^k(l_{(k)}) + (\Theta_t^{k+1}(l_{(k)}, t) - \Theta_t^k(l_{(k)}))\sigma_t^{k+1,k}(l_{(k)})\lambda_t^{k+1,k}(l_{(k)}) + \sigma_t^k(l_{(k)})\bar{\beta}_t^k(l_{(k)})\right]^2}{\Theta_t^k(l_{(k)})\sigma_t^k(l_{(k)})^2 + (\Theta_t^{(k)}(l_{(k)}) + \Theta_t^{k+1}(l_{(k)}, t) - \Theta_t^k(l_{(k)}))\sigma_t^{k+1,k}(l_{(k)})^2\lambda_t^{k+1,k}(l_{(k)})} \\
&+ \left[\Theta_t^{k+1}(l_{(k)}, t) - \Theta_t^k(l_{(k)})\right]\lambda_t^{k+1,k}(l_{(k)})
\end{aligned}$$

Let consider the processes:

$$\begin{aligned}
n_t &= \left[\mu_t^k(l_{(k)}) - \sigma_t^{k+1,k}(l_{(k)})\lambda_t^{k+1,k}(l_{(k)})\right]|\Theta_t^k(l_{(k)})| \\
\kappa_t &:= \sigma_t^{k+1,k}(l_{(k)})\lambda_t^{k+1,k}(l_{(k)})\Theta_t^{k+1}(l_{(k)}, t) \\
m_t &:= \sigma_t^k(l_{(k)})\bar{\beta}_t^k(l_{(k)})
\end{aligned}$$

and $\bar{N}_t = n_t + \kappa_t + m_t$, $d_t = |\Theta_t^k(l_{(k)})|\sigma_t^k(l_{(k)})^2$, $p_t = \Theta_t^{k+1}(l_{(k)}, t)(\sigma_t^{k+1,k}(l_{(k)})^2\lambda_t^{k+1,k}(l_{(k)})$ and $D_t = d_t + p_t$. We define

$$\begin{aligned}
f_t^k &:= \bar{f}_t^k(|\Theta_t^k(l_{(k)})|, \bar{\beta}_t^k, l_{(k)}) \\
&= -\frac{\bar{N}_t^2}{\bar{D}_t} + \left[\Theta_t^{k+1}(l_{(k)}, t) - |\Theta_t^k(l_{(k)})|\right]\lambda_t^{k+1,k}(l_{(k)})
\end{aligned} \tag{3.44}$$

where $\bar{N}_t^2 = n_t^2 + m_t^2 + \kappa_t^2 + 2n_tm_t + 2\kappa_tn_t + 2\kappa_tm_t$. Since the process $\Theta_t^{k+1}(l_{(k)}, t) \geq \delta^{k+1} > 0$ then there exists a non negative constant $c > 0$ such that $p_t > c$. Hence, we obtain:

$$\begin{aligned}
-f_t^k &:= \frac{\bar{N}_t^2}{\bar{D}_t} - \left[\Theta_t^{k+1}(l_{(k)}, t) - |\Theta_t^k(l_{(k)})|\right]\lambda_t^{k+1,k}(l_{(k)}) \\
&\leq \left[\frac{n_t^2}{d_t} + \frac{2n_t\kappa_t}{p_t} + |\Theta_t^k(l_{(k)})|\lambda_t^{k+1,k}(l_{(k)})\right] + \frac{m_t^2}{d_t} \\
&+ \left[\frac{2n_tm_t}{d_t} + \frac{2m_t\kappa_t}{p_t}\right] + \left[\frac{\kappa_t^2}{p_t} - \Theta_t^{k+1}(l_{(k)}, t)\lambda_t^{k+1,k}(l_{(k)})\right]
\end{aligned}$$

Therefore since all processes Θ^{k+1}, μ^k, $\sigma^{k+1,k}$ and $\lambda^{k+1,k}$ are bounded, there exist bounded processes a, b and c such that:

$$-f_t^k \leq h_t := a_t|\Theta_t^k(l_{(k)})| + b_t\bar{\beta}_t^k(l_{(k)}) + \frac{\bar{\beta}_t^k(l_{(k)})^2}{|\Theta_t^k(l_{(k)})|}.$$

The coefficient f^k has a quadratic growth and from Kobylanski [19] since the terminal condition γ^k is bounded, there exists a pair $(\Theta^k(l_{(k)}), \bar{\beta}(l_{(k)}))$ solution of the BSDE associated to $(f^k(l_{(k)}), \gamma^k(l_{(k)}))$. Moreover if we consider the BSDE $d\bar{x}_t = -\bar{h}(\bar{x}_t, \bar{Z}_t)dt + \bar{Z}_t dW_t$ with terminal condition $x_T = \gamma^1(l_{(1)}) = 1$ where the coefficient $\bar{k}$ is given by :

$$\bar{h}_t(\bar{x}_t, \bar{Z}_t) = -a_t\bar{x}_t - \frac{\bar{Z}_t^2}{\bar{x}_t} - b_t\bar{Z}_t, \quad t \in [0,T]$$

From Proposition 5.11 in [23], the solution $(\bar{x}, \bar{Z}) \in \mathcal{S}^\infty[0,T] \times \text{BMO}$ exists. Moreover there exists a non negative constant δ^k such that $\bar{x}_t \geq \delta^k, a.s.$ Hence we conclude $f^k \geq -\bar{h} = h$ and using Comparison theorem of quadratic BSDE (see [19]). The pair of solution $(\Theta^k(l_{(k)}), \bar{\beta}^k(l_{(k)}))$ associated to $(f^k(l_{(k)}), \gamma^k(l_{(k)}))$ satisfies $\Theta_t^k(l_{(k)}) \geq \delta^k > 0$ a.s for all $t \in [0,T], l_{(k)} \in \Delta_k$. Therefore from 3.44, we conclude $f^k = \bar{f}^k$ and it follows that there exists a solution $(\Theta^k(l_{(k)}), \bar{\beta}^k(l_{(k)})) \in \mathcal{S}^\infty[l_k,T] \times \text{BMO}$ associated to $(\bar{f}^k(l_{(k)}), \gamma^k(l_{(k)}))$ such that $\Theta_t^k(l_{(k)}) \geq \delta^k > 0$ a.s for all $t \in [0,T], l_{(k)} \in \Delta_k$.

3.5 Special case and explicit solution of the BSDE

We conclude by giving an explicit example of our credit risk model which allow us to find explicit solution of each BSDEs. We assume $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^A$ and that the parameters of the asset's dynamics are constant before and after the default time τ^A. Moreover, we assume that the intensity process is given by $\lambda_t = \lambda(1 - H_t^A)$. Using theses assumptions, we find an explicit solution of the BSDE associated to (g^1, Θ_T) using the splitting approach.

Assumption 3.4. *The processes $\mu, \sigma, \sigma^A, \lambda$ in (1.4) satisfy the following sassumptions:*

$$\begin{aligned}
\mu_t &= \mu(H_t^A) = \mu^0 1_{\{\tau^A > t\}} + \mu^1 1_{\{\tau^A \leq t\}}, \\
\sigma_t &= \sigma(H_t^A) = \sigma^0 1_{\{\tau^A > t\}} + \sigma^1 1_{\{\tau^A \leq t\}}, \\
\sigma_t^A &= \sigma^A(H_t^A) = \kappa 1_{\{\tau^A > t\}}, \\
\lambda_t &= \lambda(H_t^A) = \lambda 1_{\{\tau^A > t\}}.
\end{aligned}$$

such that $\mu^0\kappa = (\sigma^0)^2 + \kappa^2\lambda$.

Proposition 3.8. *Under Assumption 3.4, there exists a solution of the BSDE associated to (g^1, Θ) given by:*

$$\Theta_t = \exp\left[-\frac{\mu^0}{\kappa}(T-t)\right] 1_{\{\tau^A > t\}} + \exp\left[-\left(\frac{\mu^1}{\sigma^1}\right)^2 (T-t)\right] 1_{\{\tau^A \leq t\}}, \quad t \leq T.$$

Proof. Let first recall that using the splitting approach developed by [18], we can write the BSDE before and after the default. We obtain

$$\begin{aligned}
\Theta_t &= \Theta_t^0 1_{\{t < \tau^A\}} + \Theta_t^1(\tau^A) 1_{\{\tau^A \leq t\}} \\
g_t^1 &= g_t^{1,0} 1_{\{t < \tau^A\}} + g_t^{1,1} 1_{\{\tau^A \leq t\}}
\end{aligned}$$

where Θ^0 and Θ^1 satisfy the following dynamics:

$$-\frac{d\Theta_t^0}{\Theta_t^0} = g_t^{1,0}(\Theta_t^0, \theta_t^A, \beta_t^0)dt - \beta_t^0 dW_t + \lambda\theta_t^A dt, \quad \Theta_T^0 = 1,$$
$$-\frac{d\Theta_t^1(l)}{\Theta_t^1(l)} = g_t^{1,1}(\Theta_t^1(l), 0, \beta_t^1(l))dt - \beta_t^1(l)dW_t, \quad \Theta_T^1(l) = 1$$

with

$$g_t^{1,0}(\Theta_t^0, \theta_t^A, \beta_t^0) = -\frac{\left[\mu^0 + \theta_t^A\kappa\lambda + \sigma^0\beta_t^0\right]^2}{(\sigma^0)^2 + (1+\theta_t^A)\kappa^2\lambda} \quad \text{and} \quad g_t^{1,1}(\Theta_t^1(l), 0, \beta_t^1(l)) = -\frac{\left[\mu^1 + \sigma^1\beta_t^1(l)\right]^2}{(\sigma^1)^2}$$

where $l \in \Delta_1$ and $\Theta_t^1(t) - \Theta_t^0 = \theta_t^A\Theta_t^0$, see proof of Theorem 3.4 for more details. Using Assumption 3.4, setting $\beta^1(l) = 0$, we find that $g_t^{1,1}(\Theta_t^1, 0, \beta_t^1(l)) = -\left(\frac{\mu^1}{\sigma^1}\right)^2$. Since $\Theta_T^1(l) = 1$, then $\Theta^1(l) = \Theta^1$ and we get:

$$\Theta_t^1 = \exp\left[-\left(\frac{\mu^1}{\sigma^1}\right)^2 (T-t)\right], \quad t \leq T.$$

To find the solution of the first one BSDE, we set $\beta^0 = 0$ and from Assumption 3.4 we obtain

$$\mu^0\kappa = (\sigma^0)^2 + \kappa^2\lambda$$

We deduce $g_t^{1,0}(\Theta_t^0, \theta_t^A, \beta_t^0) = -\frac{\mu^0}{\kappa} - \theta_t^A\lambda$. Therefore we find that Θ^0 satisfies the dynamics:

$$-\frac{d\Theta_t^0}{\Theta_t^0} = -\frac{\mu^0}{\kappa}dt, \quad \Theta_T^0 = 1.$$

Finally, we get $\Theta_t^0 = \exp\left[-\frac{\mu^0}{\kappa}(T-t)\right]$ and we find the expected result. □

Appendix

Lemma 3.1. *Let consider X and Y two $\mathbb{G}$-predictable processes such that for $i \in \{A, B\}$, $Y_{\tau_i} = X_{\tau_i}$. Then, $X_t = Y_t$ on $(\tau_i \geq t)$ a.s. Moreover, if $X_{\tau_i} \leq Y_{\tau_i}$, then $X_t \leq Y_t$ a.s on $(\tau_i \geq t)$.*

Proof. Assume that X and Y are bounded. If $X_{\tau_i} = Y_{\tau_i}$, then $\int_0^\infty |X_t - Y_t| dH_t^i = 0$ and

$$0 = \mathbb{E}\left(\int_0^\infty |X_t - Y_t| dH_t^i\right) = \mathbb{E}\left[\int_0^\infty |X_t - Y_t|\lambda_t^i dt\right].$$

Therefore, we have $X_t = Y_t$ on $(\tau^i \geq t)$. Moreover, if $X_{\tau_i} \leq Y_{\tau_i}$, we consider the predictable process V defined as $V_t = Y_t 1_{\{X_t \leq Y_t\}}$. Then $V_{\tau^i} = Y_{\tau^i}$ and by using the first part of the proof, we obtain $V_t = Y_t$ on $(\tau^i \geq t)$. The general case follows. □

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